บันทึกข้อความ

ส่วนงาน คณะวิทยาศาสตร์ หลักสูตรวิทยาศาสตรบัณฑิต สาขาวิชาคณิตศาสตร์ โทร. 3881

ที่ ศธ 0523.4.5 / 309 **วันที่** 28 สิงหาคม 2561

เรื่อง ขอส่งสรุปเนื้อหาและการนำไปใช้ประโยชน์จากการเดินทางไปทำวิจัย ณ ต่างประเทศ

เรียน คณบดีคณะวิทยาศาสตร์

ตามหนังสือที่ ศธ 0523.4.5/ 114 ลงวันที่ 25 เมษายน 2561 ได้อนุญาตให้ข้าพเจ้า ผู้ช่วยศาสตราจารย์ ดร.ดารา ภูสง่า บุคลากรสังกัดสาขาวิชาคณิตศาสตร์ คณะวิทยาศาสตร์ มหาวิทยาลัยแม่โจ้ เดินทางไปทำวิจัย ณ ต่างประเทศ เพื่อทำวิจัย ณ คณะคณิตศาสตร์ มหาวิทยาลัยพอทสดัม (Potsdam University) ประเทศเยอรมนี โดยทำวิจัยในหัวข้อ **"เซมิกรุปของเทอมเชิงเส้นและฟอร์มูล่าเชิงเส้น (On a Semigroup of linear-terms and linear-formulas)** ระหว่างวันที่ 10 พฤษภาคม 2561-7 มิถุนายน 2561 และคณะคณิตศาสตร์และวิทยาศาสตร์ธรรมชาติ South-West University "Neofit Rilski" ประเทศบัลแกเรีย โดยทำวิจัยในหัวข้อ **"ออร์เดอร์ของไฮเพอร์สับสติติวชันเชิงเส้นสำหรับระบบพีชคณิตชนิด ((2);(2)) (Orders of Linear Hypersubstitutions for algebraic systems of type ((2);(2)))** ในระหว่างวันที่ 7-14 มิถุนายน 2561 ทั้งนี้ ข้าพเจ้าได้ออกเดินทางจากประเทศไทยวันที่ 9 พฤษภาคม 2561 และเดินทางกลับมาถึงประเทศไทยวันที่ 16 มิถุนายน 2561 บัดนี้ การเดินทางไปทำวิจัย ณ ต่างประเทศดังกล่าวได้เสร็จสิ้นเรียบร้อยแล้ว ดังนั้น ข้าพเจ้าจึงขอส่งสรุปเนื้อหาและการนำไปใช้ประโยชน์จากการเดินทางไปทำวิจัย ณ ต่างประเทศ ดังรายละเอียดที่แนบมาพร้อมนี้

จึงเรียนมาเพื่อโปรดทราบ

(ผู้ช่วยศาสตราจารย์ ดร.ดารา ภูสง่า)
อาจารย์ประจำหลักสูตรสาขาวิชาคณิตศาสตร์

รายงานสรุปเนื้อหาและการนำไปใช้ประโยชน์จากการเดินทางไปทำวิจัย ณ ต่างประเทศ

ข้าพเจ้านางดารา ภูสง่า ตำแหน่ง ผู้ช่วยศาสตราจารย์ สังกัด หลักสูตร วท.บ. สาขาวิชาคณิตศาสตร์ คณะวิทยาศาสตร์ ขอนำเสนอรายงานสรุปเนื้อหาและการนำไปใช้ประโยชน์จากการ **เดินทางไปทำวิจัย ณ ต่างประเทศ** เมื่อวันที่ 9 พฤษภาคม 2561-16 มิถุนายน 2561 ณ คณะคณิตศาสตร์ มหาวิทยาลัยพอทสดัม ประเทศเยอรมนี และ คณะคณิตศาสตร์และวิทยาศาสตร์ธรรมชาติ มหาวิทยาลัย South-West "Neofit Rilski" ประเทศบัลกาเรีย ตามหนังสือขออนุมัติเดินทางไปราชการ เลขที่ ศธ.0523.4. 5/ 114 ลงวันที่ 25 เมษายน 2561 ซึ่งการเดินทางไปทำวิจัย ณ ต่างประเทศดังกล่าว ข้าพเจ้าได้เบิกจ่ายจากแหล่งงบประมาณ ดังนี้

1) ทุนส่วนตัว
2) ทุน โดยได้รับการสนับสนุน ค่าที่พักและค่าเดินทางของมหาวิทยาลัย South-West "Neofit Rilski" จากประเทศเยอรมนีไปยังบัลกาเรีย

ดังนั้นจึงขอนำเสนอสรุปเนื้อหาและการนำไปใช้ประโยชน์ของการไปทำวิจัยดังต่อไปนี้

1. ข้าพเจ้าได้แลกเปลี่ยนความรู้ทางวิชาการด้านคณิตศาสตร์ ในหัวข้อ เรื่อง " Linear-terms and linear-formulas" กับ Prof. Dr. Koppitz ซึ่งเป็นผู้เชี่ยวชาญในหัวข้อนี้
2. หลังจากได้แลกเปลี่ยนความรู้ในหัวข้อดังกล่าวแล้ว ข้าพเจ้าได้ทำจัยร่วม กับ Prof. Dr. Koppitz ในหัวข้อ "The semigroup of linear terms"
3. ข้าพเจ้าได้รับเชิญจาก คณะคณิตศาสตร์ มหาวิทยาลัยพอทสดัม ให้บรรยายในหัวข้อเรื่อง

- Linear-terms and Empty terms
- Idempotent and regular elements
- Green's relations
- Linear-formulas over set of linear terms

4. ข้าพเจ้าได้รับเชิญจาก สถาบันวิจัยวิทยาศาสตร์แห่งชาติบัลการเรีย(Institute of Mathematics and Informatics Bulgarian Academy of Sciences)บรรยายในหัวข้อเรื่อง "The order of hypersubstitutions for algebraic systems of type ((2);(2))"

การนำไปใช้ประโยชน์

1. จากการไปทำวิจัยในครั้งนี้ ได้พบปะและแลกเปลี่ยนแนวคิดทำให้มองเห็นแนวทางการทำงานวิจัยในอนาคต อีกทั้งเห็นแนวทางในการพัฒนาการเรียนการสอนในหลักสูตรที่รับผิดชอบ และแนวทางการสร้างความร่วมมือทางวิชาการจากบุคคลากรของสถาบันการศึกษาซึ่งเป็นประโยชน์โดยตรงกับสายงานของผู้รายงาน
2. ข้าพเจ้าได้นำความรู้ที่ไปทำวิจัยในครั้งนี้ มาช่วยในการพิสูจน์ทฤษฎีที่สำคัญได้ จนนำไปสู่การเขียนงานวิจัยที่สมบูรณ์ เรื่อง "The semigroup of linear terms" ได้รับการตีพิมพ์เรียบร้อยในวราสาร **Asian-European Journal of Mathematics**, Volume 13, No. 1 ที่อยู่ในฐาน Scopus

....

(นางดารา ภูสง่า)

ความคิดเห็นของผู้บังคับบัญชาชั้นต้น (ประธานหลักสูตร/เลขานุการคณะ/หัวหน้างาน)

เป็นกิจกรรมที่มีผลสัมฤทธิ์เป็นรูปธรรม ซึ่งสามารถพัฒนาตนเอง

และคาดว่าเป็นประโยชน์ในการพัฒนาหลักสูตรต่อไป

....

(ผู้ช่วยศาสตราจารย์จินตนา จูมวงษ์)

ประธานหลักสูตร วท.บ. สาขาวิชาคณิตศาสตร์

ความคิดเห็นของคณบดีคณะวิทยาศาสตร์หรือผู้แทน

....

....

....

(..)

....

Asian-European Journal of Mathematics
Vol. 13, No. 1 (2020) 2050005 (9 pages)

DOI: 10.1142/S1793557120500059

The semigroup of linear terms

D. Phusanga*

Department of Mathematics, Faculty of Science
Maejo University, Chiang Mai 50290, Thailand
darapu@mju.ac.th

J. Koppitz

Institute of Mathematics and Informatics
Bulgarian Academy of Sciences, Sofia 1113, Bulgaria
koppitz@math.bas.bg

Communicated by I. Chajda
Received April 2, 2018
Revised June 11, 2018
Accepted June 16, 2018
Published July 19, 2018

The set of linear terms, i.e. terms in which each variable occurs at most once, does not form a subsemigroup of the so-called diagonal semigroup. We consider the reduct of the diagonal semigroup to the linear terms, which is not a partial semigroup. We extend the set of linear terms by an expression "∞", that is formally a linear term, obtaining a semigroup. The algebraic structure of this semigroup will be studied in this paper. We characterize the Green's relations and the regular elements as well as the idempotent elements. Moreover, we discuss the ideal structure.

Keywords: Linear terms; diagonal semigroup; Green's relations; regular elements; ideals.

AMS Subject Classification: 08A30, 08A55

1. Introduction

In algebra, an important problem is the study of the algebraic structure of a given universal algebra. A classical class of universal algebras is the variety of semigroups [1]. Semigroups are important structures in Algebra and also in other fields of mathematics (e.g. in theoretical computer sciences: the word semigroup). In fact, a semigroup is a pair consisting of a nonempty set (the universe) and an associative operation on this set. The structure of a semigroup can be characterized by Green's relations very well [7]. Moreover, the regular and the idempotent elements of a semigroup as well as its ideal structure are of particular interest.

*Corresponding author.

In the present paper, we study a semigroup, whose universe is a particular subset of the set of all terms of a given type over a countable infinite alphabet. A semigroup of terms of a given type, the so-called diagonal semigroup, was first studied by Denecke and Jampachon in 2006 [4]. Concerning term functions, the concept of so-called linear terms appears (e.g. [2]). Recall that a term over a given alphabet, containing each variable at most once, is called linear. Note that there are also other names for such terms. For the algebraic background of linear terms, we refer the reader to the paper [3] by K. Denecke.

In [8], the authors define a partial binary operation on the set of linear terms of type $\tau = (n, \ldots, n)$, for a given natural number n, and study the regular elements as well as the Green's relations. It will turn out that this partial operation is not associative. So, one has to question the results in [8]. In the present paper, we want to correct this situation, by introducing a semigroup which extends the partial operation. Since the partial operation is not associative, the one-point extension is not practicable. We will consider another extension. For this, we introduce an expression "∞" which is characterized in that it contains no variables and no operation symbol. In particular, ∞ can be regarded as linear since it has no variable and so it satisfies the definition of linear term formally.

In the next section, we summarize preliminary definitions, notations, and facts. In particular, we introduce an extension of the partial operation given in [8] and show that it is associative. In fact, we introduce a semigroup of linear terms. In the rest of the paper, we study the algebraic structure of this semigroup. The third section is devoted to the idempotent and the regular elements as well as the ideal structure. In the last section, we characterize all the Green's relations.

2. Preliminaries

Let $X := \{x_1, x_2, \ldots\}$ be a countable infinite set, which is called alphabet and its elements are called variables. We fix a set $\{f_i : i \in I\}$ of operation symbols for an index set I. To every operation symbol $f_i (i \in I)$, we assign a natural number $n_i \geq 1$, which is called the arity of f_i. The sequence $\tau = (n_i)_{i \in I}$ will be called type. Let $n \geq 1$ be a natural number and let $W_\tau(X_n)$ be the smallest set containing the variables $x_1, \ldots, x_n$ and being closed under the following operation: if $i \in I$ and $t_1, \ldots, t_{n_i} \in W_\tau(X_n)$ then $f_i(t_1, \ldots, t_{n_i}) \in W_\tau(X_n)$. The elements in $W_\tau(X_n)$ are called n-ary terms of type τ. Note that every n-ary term is also k-ary, whenever $k > n$. Denote by $W_\tau(X)$ the set of all terms of type τ over X, i.e. $W_\tau(X) := \bigcup_{n \in \mathbb{N}} W_\tau(X_n)$, where $\mathbb{N}$ denotes the set of natural numbers. Throughout this paper, we will use the following notations for a term t of type τ:

$c(t)$ is the set of all variables occurring in t;
$c_i(t)$ is the number of occurrence of the variable x_i in t, for $i \in \mathbb{N}$;
$op(t)$ is the number of operation symbols occurring in t.

Now, we will fix a natural number n. For any natural number m, the superposition S^n_m of an n-ary term with n m-ary terms $t_1, \ldots, t_n$ is defined as following:

$$S^n_m(x_i, t_1, \ldots, t_n) := t_i \quad \text{for } 1 \leq i \leq n \text{ and}$$

$S^n_m(f_i(s_1, \ldots, s_{n_i}), t_1, \ldots, t_n) := f_i(S^n_m(s_1, t_1, \ldots, t_n), \ldots, S^n_m(s_{n_i}, t_1, \ldots, t_n))$ for $i \in I$ and $s_1, \ldots, s_{n_i} \in W_\tau(X_n)$ [6]. The operation S^n_m has a very nice formal interpretation. In fact, $S^n_m(s, t, \ldots, t)$ is nothing else than the term s but all the occurrences of the variables $x_1, \ldots, x_n$ are replaced by the term t, for any (linear) terms $s, t \in W_\tau(X)$. Denecke and Leeratanavalee introduced a generalized superposition of a k-ary term with n m-ary terms $t_1, \ldots, t_n$ for any $k \in \mathbb{N}$ [5], i.e. they defined $S^n_m(t, t_1, \ldots, t_n)$ for the case that $t \notin W_\tau(X_n)$. In fact, they put $S^n_m(x_i, t_1, \ldots, t_n) := x_i$, whenever $i > n$. By the inductive steps, $S^n_m(t, t_1, \ldots, t_n)$ is defined for any term t. Subsequently, the natural number m in the expression $S^n_m(t, t_1, \ldots, t_n)$ will be any suitable natural number such that $t_1, \ldots, t_n \in W_\tau(X_m)$. It was shown that S^n_m in this general setting satisfies the superassociative law [6]. In [4], Denecke and Jampachon define a binary associative operation $+_g$ on the set of terms of type τ by $s +_g t := S^n_m(s, t, \ldots, t)$, whenever $t \in W_\tau(X_m)$ and $s \in W_\tau(X)$. Note that the operation $+_g$ depends on the natural number n, which was not indicated by the authors.

Recall that a linear term of type τ is a term t with $c_i(t) \leq 1$ for all $i \in \mathbb{N}$. In [8], the authors define a partial binary operation $+_n$ on the set of linear terms by

$$s +_n t := \begin{cases} S^n_m(s, t, \ldots, t) & \text{if } S^n_m(s, t, \ldots, t) \text{ is linear} \\ \text{not defined} & \text{otherwise.} \end{cases}$$

We observe that $+_n$ is not associative. For this, let $r, s, t \in W_\tau(X)$ be linear terms with $c(r) \cap X_n = \emptyset$, $c_{n+1}(s) = c_{n+1}(t) = 1$, and $c(s) \cap X_n = \{x_1\}$. Then $s +_n t$ is not defined since $c_{n+1}(s +_n t) = 2$, i.e. $r +_n (s +_n t)$ is not defined. On the other hand, $(r +_n s) +_n t$ is defined since we can easy calculate that $r +_n s = r$ and $r +_n t = r$.

Let $W^{\text{lin}}_\tau(X)^\infty$ be the set of all linear terms of type τ extended to the expression ∞. Since ∞ does not contain any variable, i.e. each variable occurs formally at most once, one can regard the expression ∞ as linear. In particular, we can state that $c(\infty) = \emptyset$. We will extend the partial operation $+_n$ to $W^{\text{lin}}_\tau(X)^\infty$ obtaining a full operation by

$$s +_n t := \begin{cases} S^n_m(s, t, \ldots, t) & \text{if } s, t \neq \infty \text{ and } S^n_m(s, t, \ldots, t) \text{ is linear} \\ s & \text{if } t = \infty \text{ and } c(s) \cap X_n = \emptyset \\ \infty & \text{otherwise,} \end{cases}$$

for $s, t \in W^{\text{lin}}_\tau(X)^\infty$. We have to verify that $+_n$ is associative. First, we observe, by the definition of the operation $+_n$, each $s \in W^{\text{lin}}_\tau(X)^\infty$ with $c(s) \cap X_n = \emptyset$ is a left zero. This provides the following lemma.

Lemma 1. *Let* $s \in W_\tau^{\text{lin}}(X)^\infty$ *with* $\text{var}(s) \cap X_n = \emptyset$. *Then* $s +_n t = s$ *for all* $t \in W_\tau^{\text{lin}}(X)^\infty$.

The formal interpretation of S_m^n justifies that $s +_n t$ is not defined, whenever more than one variable of X_n occurs in s, for all (linear) terms $t \in W_\tau(X)$. This proves the following lemma.

Lemma 2. *Let* $s \in W_\tau^{\text{lin}}(X)^\infty$ *such that* $|c(s) \cap X_n| \geq 2$. *Then* $s +_n t = \infty$ *for all* $t \in W_\tau^{\text{lin}}(X)^\infty$.

The verbal definition of S_m^n justifies also the following observation.

Lemma 3. *Let* $s, t \in W_\tau(X)$ *be linear terms and let* $k \in \mathbb{N}$. *If* $|c(s) \cap X_n| = 1$ *then*

$$c_k(S_m^n(s, t, \ldots, t)) = \begin{cases} c_k(s) + c_k(t) & \text{if } k > n \\ c_k(t) & \text{if } k \leq n. \end{cases}$$

Now, we are able to verify that $+_n$ is associative.

Proposition 4. $(W_\tau^{\text{lin}}(X)^\infty; +_n)$ *is a semigroup.*

Proof. It is to show that $+_n$ is associative. We will use the previous lemmas throughout this proof without to refer them. Let $r, s, t \in W_\tau^{\text{lin}}(X)^\infty$. We have to verify that $(r +_n s) +_n t = r +_n (s +_n t)$. If $c(r) \cap X_n = \emptyset$ (it includes $r = \infty$) or $|c(r) \cap X_n| \geq 2$ then it is easy to verify that $(r +_n s) +_n t = r +_n (s +_n t) \in \{\infty, r\}$. Suppose now that $|c(r) \cap X_n| = 1$. If $|c(s) \cap X_n| \geq 2$ and thus $|c(S_m^n(r, s, \ldots, s)) \cap X_n| \geq 2$, we obtain $(r +_n s) +_n t = r +_n (s +_n t) = \infty$. If $c(s) \cap X_n = \emptyset$ (it includes $s = \infty$) and thus $c(r +_n s) \cap X_n = \emptyset$, we have $(r +_n s) +_n t = r +_n (s +_n t) = r +_n s$. Suppose now that $|c(s) \cap X_n| = 1$, i.e. $|c(S_m^n(r, s, \ldots, s)) \cap X_n| = 1$. If $S_m^n(r, s, \ldots, s)$ is not linear then there is a natural number $k > n$ with $c_k(S_m^n(r, s, \ldots, s)) = 2$, i.e. $x_k \in c(r) \cap c(s)$ as well as $x_k \in c(S_m^n(s, t, \ldots, t))$, whenever $t \neq \infty$ and $S_m^n(s, t, \ldots, t)$ is linear. This insures that $r +_n (s +_n t) = \infty$. On the other hand, we have $r +_n s = \infty$ (because $c_k(S_m^n(r, s, \ldots, s)) = 2$) and thus, $(r +_n s) +_n t = \infty$. Suppose now that $S_m^n(r, s, \ldots, s)$ is linear, i.e. $r +_n s = S_m^n(r, s, \ldots, s)$. If $t = \infty$ then $(r +_n s) +_n t = r +_n (s +_n t) = \infty$. So, we have to consider the case $t \neq \infty$. If $S_m^n(s, t, \ldots, t)$ is not linear then there is a natural number $l > n$ with $c_l(S_m^n(s, t, \ldots, t)) = 2$, i.e. $x_l \in c(s) \cap c(t)$. In particular, $x_l \in c(S_m^n(r, s, \ldots, s))$ and thus, $c_l(S_m^n(S_m^n(r, s, \ldots, s), t, \ldots, t)) = 2$. This means that $(r +_n s) +_n t = \infty$. On the other hand, we have $r +_n (s +_n t) = r +_n \infty = \infty$. It remains the case that $S_m^n(s, t, \ldots, t)$ is linear. Let $j \in \mathbb{N}$. If $j > n$ then $c_j(S_m^n(r, S_m^n(s, t, \ldots, t), \ldots, S_m^n(s, t, \ldots, t))) = c_j(r) + c_j(S_m^n(s, t, \ldots, t)) = c_j(r) + c_j(s) + c_j(t) = c_j(S_m^n(r, s, \ldots, s)) + c_j(t) = c_j(S_m^n(S_m^n(r, s, \ldots, s), t, \ldots, t))$. If $j \leq n$ then $c_j(S_m^n(r, S_m^n(s, t, \ldots, t), \ldots, S_m^n(s, t, \ldots, t))) = c_j(S_m^n(s, t, \ldots, t)) = c_j(t) = c_j(S_m^n(S_m^n(r, s, \ldots, s), t, \ldots, t))$. This shows that $S_m^n(S_m^n(r, s, \ldots, s), t, \ldots, t)$ is linear if and only if $S_m^n(r, S_m^n(s, t, \ldots, t), \ldots, S_m^n(s, t, \ldots, t))$ is linear. Hence, $(r +_n s) +_n t = r +_n (s +_n t) = \infty$ or both terms $S_m^n(S_m^n(r, s, \ldots, s), t, \ldots, t)$ and $S_m^n(r, S_m^n(s, t, \ldots, t), \ldots, S_m^n(s, t, \ldots, t))$ are linear and we have the equality $(r +_n$

$s) +_n t = S^n_m(S^n_m(r, s, \ldots, s), t, \ldots, t) = S^n_m(r, S^n_m(s, t, \ldots, t), \ldots, S^n_m(s, t, \ldots, t)) = r +_n (s +_n t)$ because of the superassociative law for S^n_m. □

3. Idempotent and Regular Elements

In Lemma 1, we have shown that $s +_n t = s$ for all $t \in W^{\text{lin}}_\tau(X)^\infty$, whenever $c(s) \cap X_n = \emptyset$. Now, we characterize all pairs $(s, t) \in (W^{\text{lin}}_\tau(X)^\infty)^2$ such that $s +_n t = s$.

Lemma 5. *Let $s, t \in W^{\text{lin}}_\tau(X)^\infty$. Then $s +_n t = s$ if and only if $c(s) \cap X_n = \emptyset$ or $s = t \in X_n$ or $s = f_i(s_1, \ldots, s_{n_i})$ is a composed term such that there is $j \in \{1, \ldots, n_i\}$ with $c(s_k) \cap X_n = \emptyset$ for all $k \in \{1, \ldots, n_i\} \backslash \{j\}$ and $c(s_j) \cap X_n = \{t\}$.*

Proof. Suppose that $s +_n t = s$ and additional suppose that $c(s) \cap X_n \neq \emptyset$ and either $s \neq t$ or $s = t \notin X_n$. From $s +_n t = s$ and $c(s) \cap X_n \neq \emptyset$, it follows $|c(s) \cap X_n| = 1$ by Lemma 2. In particular, this implies $t \neq \infty$. Assume that $s \in X_n$ but $s \neq t$. Then $s +_n t = S^n_m(s, t, \ldots, t) = t \neq s$, a contradiction. Hence, $s = t$ or $s \notin X_n$. This implies $s \notin X_n$, i.e s a composed term $s = f_i(s_1, \ldots, s_{n_i})$ with $c(s) \cap X_n \neq \emptyset$. Then there is $j \in \{1, \ldots, n_i\}$ with $c(s_k) \cap X_n = \emptyset$ for all $k \in \{1, \ldots, n_i\} \backslash \{j\}$ and $|c(s_j) \cap X_n| = 1$ because $|c(s) \cap X_n| = 1$. Assume that $s_j \notin X_n$ or $t \neq s_j$. Then $S^n_m(s_j, t, \ldots, t) \neq s_j$ by Lemma 3 and thus, $s +_n t = S^n_m(f_i(s_1, \ldots, s_{n_i}), t, \ldots, t) = f_i(S^n_m(s_1, t, \ldots, t), \ldots, S^n_m(s_{n_i}, t, \ldots, t)) \neq f_i(s_1, \ldots, s_{n_i}) = s$, a contradiction. Hence, $s_j \in X_n$ and $t = s_j$, i.e. $c(s_j) \cap X_n = \{t\}$.

The converse direction can be easy verified by the calculation of $s +_n t$ under the given conditions. □

Let E_n be the set of all idempotents elements in the semigroup $(W^{\text{lin}}_\tau(X)^\infty; +_n)$, i.e.

$$E_n := \{s \in W^{\text{lin}}_\tau(X)^\infty : s +_n s = s\}.$$

Lemma 5 provides a characterization of E_n.

Proposition 6. *$E_n = X_n \cup \{s \in W^{\text{lin}}_\tau(X)^\infty : c(s) \cap X_n = \emptyset\}$ and E_n forms a subsemigroup of $(W^{\text{lin}}_\tau(X)^\infty; +_n)$.*

Proof. By Lemma 5, we have $s +_n s = s$ if and only if $c(s) \cap X_n = \emptyset$ or $s \in X_n$ or $s = f_i(s_1, \ldots, s_{n_i})$ is a composed term such that there is $j \in \{1, \ldots, n_i\}$ with $c(s_k) \cap X_n = \emptyset$ for all $k \in \{1, \ldots, n_i\} \backslash \{j\}$ and $c(s_j) \cap X_n = \{s\}$. Since the latter case is obviously not possible (it would provide $s = f_i(s_1, \ldots, s_{n_i}) \in X_n$), the first part of the proof is done. It remains to show that E_n is closed under $+_n$. But this is the case by Lemma 1 and because $x_i +_n s = s$ for $1 \leq i \leq n$ and $s \in W^{\text{lin}}_\tau(X)^\infty$. □

Now, we determine the regular elements in $(W^{\text{lin}}_\tau(X)^\infty; +_n)$. Recall that an element $s \in W^{\text{lin}}_\tau(X)^\infty$ is called regular (in $(W^{\text{lin}}_\tau(X)^\infty; +_n)$) if there is a $t \in W^{\text{lin}}_\tau(X)^\infty$ such that $s +_n t +_n s = s$. Let Reg_n be the set of all regular elements in $(W_\tau^{\text{lin}}(X)^\infty; +_n)$. Clearly, $E_n \subseteq \text{Reg}_n$. In general, the converse inclusion is not valid. But we will obtain the equality for the semigroup $(W_\tau^{\text{lin}}(X)^\infty; +_n)$. In order to verify this, we have to prove the following fact.

Lemma 7. *Let $s, t \in W_\tau^{\text{lin}}(X)^\infty$. Then $s +_n t \in X$ if and only if $s \in X$ and $t \in X$, whenever $s \in X_n$.*

Proof. Suppose that $s +_n t \in X$. Then, we observe that $s \in X$ by Lemma 1 until Lemma 3. Suppose that $s \in X_n$. Then, we conclude $t \neq \infty$ and $s +_n t = S_m^n(s, t, \ldots, t) = t$, i.e. $t \in X$.

Conversely, we have

$$s +_n t = \begin{cases} s \in X & \text{if } s \notin X_n \\ t \in X & \text{if } s \in X_n. \end{cases} \qquad \square$$

We will use Lemmas 5 and 7 in order to show that the set of the regular elements coincides with the set of the idempotent elements in $(W_\tau^{\text{lin}}(X)^\infty; +_n)$.

Proposition 8. $E_n = \text{Reg}_n$.

Proof. We have $E_n \subseteq \text{Reg}_n$. Let now $s \in \text{Reg}_n$, i.e. there is $t \in W_\tau^{\text{lin}}(X)^\infty$ such that $s +_n (t +_n s) = s$. By Lemma 5, we have $c(s) \cap X_n = \emptyset$ or $s = t +_n s \in X_n$ (i.e. $s \in E_n$) or $s = f_i(s_1, \ldots, s_{n_i})$ is a composed term such that there is $j \in \{1, \ldots, n_i\}$ with $c(s_k) \cap X_n = \emptyset$ for all $k \in \{1, \ldots, n_i\} \backslash \{j\}$ and $c(s_j) \cap X_n = \{t +_n s\}$. The latter one is not possible. Otherwise, $t \in X \backslash X_n$ by Lemma 7 (since $t +_n s \in X$ and $s \notin X$), i.e. $t +_n s = t \notin X_n$, a contradiction. $\square$

In the last part of this section, we study the ideal structure of the semigroup $(W_\tau^{\text{lin}}(X)^\infty; +_n)$. We will show that there are countable infinite many ideals. For $p \in \mathbb{N}$, let

$$I_p := \{t \in W_\tau^{\text{lin}}(X)^\infty : op(t) \geq p\} \cup \{t \in W_\tau^{\text{lin}}(X)^\infty : c(t) \cap X_n = \emptyset\}.$$

Proposition 9. *I_p is an ideal of $(W_\tau^{\text{lin}}(X)^\infty; +_n)$, for all $p \in \mathbb{N}$.*

Proof. We have to show that both terms $s +_n t$ and $t +_n s$ belong to $W_\tau^{\text{lin}}(X)^\infty$, for each $s \in W_\tau^{\text{lin}}(X)^\infty$ and each $t \in I_p$.

If $c(t) \cap X_n = \emptyset$ then $t +_n s = t \in I_p$. Further, we have that either $s +_n t = \infty \in I_p$ or $s +_n t$ is a linear term with $c(s +_n t) \cap X_n = \emptyset$, i.e. $s +_n t \in I_p$.

Now, we consider the case that $|c(t) \cap X_n| \geq 1$ and $op(t) \geq p$. Then either $s = \infty$ or $op(S_m^n(t, s, \ldots, s)) \geq op(t) \geq p$ by the verbal definition of S_m^n. Hence, either $t +_n s = \infty$ or $op(t +_n s) \geq p$. On the other hand, we have that either $c(s) \cap X_n = \emptyset$ or $op(S_m^n(s, t, \ldots, t)) \geq op(t) \geq p$ by the verbal definition of S_m^n. Therefore, $s +_n t = s \in \{r \in W_\tau^{\text{lin}}(X)^\infty : c(r) \cap X_n = \emptyset\}$ (it includes $s +_n t = \infty$) or $op(s +_n t) \geq p$, i.e. $s +_n t \in I_p$. $\square$

Proposition 9 shows that there are countable infinite many ideals of $(W_\tau^{\text{lin}}(X)^\infty; +_n)$. It is easy to verify that $I_1 = W_\tau^{\text{lin}}(X)^\infty \backslash X_n$. In fact, we have $t \in W_\tau^{\text{lin}}(X)^\infty \backslash X_n$ if and only if $t = \infty$ or $t \in X \backslash X_n$ or $op(t) \geq 1$ if and only if $t \in I_1$ since $X \backslash X_n \subseteq \{r \in W_\tau^{\text{lin}}(X)^\infty : c(r) \cap X_n = \emptyset\}$.

Proposition 10. *I_1 is the greatest proper ideal of $(W_\tau^{\text{lin}}(X)^\infty, +_n)$.*

Proof. Let $I \neq W_\tau^{\text{lin}}(X)^\infty$ be an ideal of $(W_\tau^{\text{lin}}(X)^\infty; +_n)$. Assume that $I \nsubseteq I_1$. Then there is $i \in \{1, \ldots, n\}$ with $x_i \in I$. Because of $W_\tau^{\text{lin}}(X)^\infty = \{x_i + t : t \in W_\tau^{\text{lin}}(X)^\infty\} \subseteq I$, we obtain $I = W_\tau^{\text{lin}}(X)^\infty$, a contradiction. □

4. The Green's Relations

In this section, we characterize the Green's relations $\mathcal{R}, \mathcal{L}, \mathcal{H}, \mathcal{D}$ and $\mathcal{J}$ for the semigroup $(W_\tau^{\text{lin}}(X)^\infty; +_n)$. Let us recall the definitions of these five equivalence relations. Since $(W_\tau^{\text{lin}}(X)^\infty; +_n)$ does not form a monoid, we consider the corresponding monoid $((W_\tau^{\text{lin}}(X)^\infty)^1; +_n)$ with the identity element 1. Let $s, t \in W_\tau^{\text{lin}}(X)^\infty$. Then

$s\mathcal{R}t$ if there are $x, y \in (W_\tau^{\text{lin}}(X)^\infty)^1$ such that $s +_n x = t$ and $t +_n y = s$;
$s\mathcal{L}t$ if there are $x^*, y^* \in (W_\tau^{\text{lin}}(X)^\infty)^1$ such that $x^* +_n s = t$ and $y^* +_n t = s$;
$s\mathcal{J}t$ if there are $x, x^*, y, y^* \in (W_\tau^{\text{lin}}(X)^\infty)^1$ such that $x^* +_n s +_n x = t$ and $y^* +_n t +_n y = s$.

The relations $\mathcal{H}$ is the intersection of the relations $\mathcal{R}$ and $\mathcal{L}$, i.e. $\mathcal{H} = \mathcal{R} \cap \mathcal{L}$. Finally, $\mathcal{D}$ is the product of $\mathcal{R}$ and $\mathcal{L}$, i.e. $\mathcal{D} = \mathcal{R} \circ \mathcal{L} = \mathcal{L} \circ \mathcal{R}$, where $\mathcal{R} \circ \mathcal{L} = \{(s, t) : \exists u \in W_\tau^{\text{lin}}(X)^\infty$ such that $(s, u) \in \mathcal{R}$ and $(u, t) \in \mathcal{L}\}$. It is well known that $\mathcal{R} \cup \mathcal{L} \subseteq \mathcal{D} \subseteq \mathcal{J}$.

First, we will prove that $s\mathcal{R}t$ if $s = t$ or s differs from t only in one variable in X_n. We mean that s differs from t only in one variable in X_n if s can be obtained from t by replacing only one variable from X_n in s (let say x) by another variable from X_n in t (let say y). One can easy verify that it is equivalent to $s +_n y = t$ and $t +_n x = s$.

Proposition 11. *Let $s, t \in W_\tau^{\text{lin}}(X)^\infty$. Then $s\mathcal{R}t$ if and only if $s = t$ or $|c(s) \cap X_n| = 1$ and s differs from t only in one variable in X_n.*

Proof. Suppose that $|c(s) \cap X_n| = 1$ and s differs from t only in one variable in X_n. Then $|c(t) \cap X_n| = 1$ and there are $x, y \in X_n$ such that $c(s) \cap X_n = \{x\}$ and $c(t) \cap X_n = \{y\}$. We can conclude that $s\mathcal{R}t$.

Suppose now that $s\mathcal{R}t$. Then there are $x, y \in (W_\tau^{\text{lin}}(X)^\infty)^1$ such that $s +_n x = t$ and $t +_n y = s$, i.e. $s = s +_n (x +_n y)$. If $x = 1$ or $y = 1$ then $s = t$. Suppose now that $x \neq 1$ and $y \neq 1$. By Lemma 5, we have $c(s) \cap X_n = \emptyset$ or $s = x +_n y \in X_n$ or $s = f_i(s_1, \ldots, s_{n_i})$ is a composed term such that there is $j \in \{1, \ldots, n_i\}$ with $c(s_k) \cap X_n = \emptyset$ for all $k \in \{1, \ldots, n_i\} \backslash \{j\}$ and $c(s_j) \cap X_n = \{x +_n y\}$. Note that $c(s) \cap X_n = \emptyset$ and $s +_n x = t$ imply $s = t$ (by Lemma 1). In the other both cases, we can calculate that $|c(s) \cap X_n| = 1$ and $x +_n y \in X_n$. Clearly, $x +_n y \in X_n$ implies

$x, y \in X_n$ by Lemmas 1 and 7. Now, we obtain that s differs from t only in one variable in X_n. This finishes the proof. □

In order to describe the $\mathcal{L}$-relation for $(W_\tau^{\text{lin}}(X)^\infty; +_n)$, we characterize the pairs $(s,t) \in (W_\tau^{\text{lin}}(X)^\infty)^2$ with $s +_n t = t$. Note that $op(s +_n t) \geq op(t)$ for all $s, t \in W_\tau^{\text{lin}}(X)^\infty$ with $c(s) \cap X_n \neq \emptyset$ by the verbal definition of S_m^n and the fact that $s +_n \infty = \infty$, whenever $c(s) \cap X_n \neq \emptyset$.

Lemma 12. *Let $s, t \in W_\tau^{\text{lin}}(X)^\infty$. Then $s +_n t = t$ if and only if $s \in X_n$ or $s = t$ with $c(t) \cap X_n = \emptyset$.*

Proof. One direction is clear. Suppose that $s +_n t = t$. Admit now that $s \notin X_n$, i.e. $s = \infty$ or $s \in X \backslash X_n$ or $op(s) \geq 1$. If $s = \infty$, or $s \in X \backslash X_n$ then $s = s +_n t = t$, where $c(t) \cap X_n = \emptyset$, i.e $s = t$ with $c(t) \cap X_n = \emptyset$. Consider now the case $op(s) \geq 1$. Assume that $c(s) \cap X_n \neq \emptyset$. Then $op(t) = op(s +_n t) > op(t)$, a contradiction. Therefore, $c(s) \cap X_n = \emptyset$ and $s +_n t = s$, i.e. $s = t$. □

Proposition 13. *Let $s, t \in W_\tau^{\text{lin}}(X)^\infty$. Then $s\mathcal{L}t$ if and only if $s = t$ or $c(s), c(t) \subseteq X \backslash X_n$.*

Proof. If $c(s), c(t) \subseteq X \backslash X_n$ then $s +_n t = s$ and $t +_n s = t$ by Lemma 1, i.e. $s\mathcal{L}t$.

Suppose now that $s\mathcal{L}t$. Then there are $x, y \in (W_\tau^{\text{lin}}(X)^\infty)^1$ such that $x +_n s = t$ and $y +_n t = s$. In particular, we get $(y +_n x) +_n s = s$. If $y = 1$ or $x = 1$ then we have $s = t$. Admit now that $x, y \neq 1$. By Lemma 12, we obtain $y +_n x \in X_n$ or $y +_n x = s$ with $c(s) \cap X_n = \emptyset$. Suppose that $y +_n x \in X_n$. It is easy to verify by Lemmas 1 and 7 that $x, y \in X_n$, and thus $s = y +_n t = t$. Suppose that $y +_n x = s$ with $c(y +_n x) \cap X_n = \emptyset$. But $c(y +_n x) \cap X_n = \emptyset$ implies $s = (y +_n x) +_n s = (y +_n x)$ by Lemma 1, which provides $c(s) = c(y +_n x) \subseteq X \backslash X_n$. Dually, we can show that $s = t$ or $c(t) = c(x +_n y) \subseteq X \backslash X_n$. This completes the proof. □

Propositions 11 and 13 provide the Green's relation $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$. It is the diagonal on $W_\tau^{\text{lin}}(X)^\infty$.

Corollary 14. $\mathcal{H} = \{(s, s) : s \in W_\tau^{\text{lin}}(X)^\infty\}$.

Proof. Let $s, t \in W_\tau^{\text{lin}}(X)^\infty$. It holds $s\mathcal{H}t$ if and only if $s = t$ or both $c(s) \subseteq X \backslash X_n$ (Proposition 13) and $|c(s) \cap X_n| = 1$ (Proposition 11). Since the latter one is not possible, we conclude that $s\mathcal{H}t$ if and only if $s = t$. □

In particular, Corollary 14 shows that the $\mathcal{H}$-classes are singleton sets. It remains to determine the $\mathcal{J}$-relation (and the $\mathcal{D}$ -relation). It is clear that $\mathcal{L} \cup \mathcal{R} \subseteq \mathcal{J}$, but for our semigroup, we have the equality.

Proposition 15. $\mathcal{J} = \mathcal{L} \cup \mathcal{R}$.

Proof. Since $\mathcal{L} \cup \mathcal{R} \subseteq \mathcal{J}$, it remains to show the converse inclusion. For this let $s, t \in W_\tau^{\text{lin}}(X)^\infty$ with $s\mathcal{J}t$. Then there are $x, x^*, y, y^* \in (W_\tau^{\text{lin}}(X)^\infty)^1$ such that

$x^* +_n s +_n x = t$ and $y^* +_n t +_n y = s$. If $x^* = y^* = 1$ then $s\mathcal{R}t$. Suppose now that $x^* \neq 1$ or $y^* \neq 1$. Without loss of generality let $x^* \neq 1$. If $c(t) \cap X_n = \emptyset$ then $t +_n y = t$ and $c(t +_n y) \cap X_n = \emptyset$. By Lemma 3, we obtain $c(y^* +_n (t +_n y)) \cap X_n = \emptyset$, i.e. $c(s) \cap X_n = \emptyset$. So, Proposition 13 provides $s\mathcal{L}t$. Dually, we can conclude $s\mathcal{L}t$ from $c(s) \cap X_n = \emptyset$.

Suppose now that $c(s) \cap X_n \neq \emptyset$ and $c(t) \cap X_n \neq \emptyset$. Then $x^*, y^* \neq \infty$ and $x^* +_n s, y^* +_n t \neq \infty$, i.e. both terms $x^* +_n s = S^n_m(x^*, s, \ldots, s)$ and $y^* +_n t = S^n_m(y^*, t, \ldots, t)$ are linear, and moreover, we have $|c(S^n_m(x^*, s, \ldots, s)) \cap X_n| = |c(S^n_m(y^*, t, \ldots, t)) \cap X_n| = 1$ by Lemma 2. This implies $x, y \neq \infty$ and $|c(x^*) \cap X_n| = |c(y^*) \cap X_n| = 1$ by Lemmas 1 and 2. Assume that $x^* \notin X_n$ or $y^* \notin X$. Without loss of generality, let $x^* \notin X_n$. Then there is a natural number $k > n$ with $x_k \in c(x^*)$. By Lemma 3, we obtain $x_k \in c(S^n_m(S^n_m(x^*, s, \ldots, s), x, \ldots, x)) = c(t)$ and thus, $x_k \in c(S^n_m(S^n_m(y^*, t, \ldots, t), y, \ldots, y)) = c(s)$. Hence, $c_k(S^n_m(x^*, s, \ldots, s)) = 2$ by Lemma 3, too, a contradiction to the linearity of $S^n_m(x^*, s, \ldots, s)$. Therefore, $x^*, y^* \in X_n$ and we conclude $t = x^* +_n s +_n x = s +_n x$ and $s = y^* +_n t +_n y = t +_n y$, i.e. $s\mathcal{R}t$. □

It is easy to verify that the Green's relation D coincides with J. In fact, we have $\mathcal{L} \cup \mathcal{R} \subseteq \mathcal{D} \subseteq \mathcal{J} = \mathcal{L} \cup \mathcal{R}$, i.e. $D = \mathcal{L} \circ \mathcal{R} = \mathcal{J}$.

References

1. A. H. Clifford and G. B. Preston, The algebraic theory of semigroups, *Amer. Math. Soc.* (1961).
2. M. Couceiro and E. Lehtonen, Galois theory for sets of operations closed under permutations, cylindrification and composition, *Algebra Universalis* **67** (2012) 273–297.
3. K. Denecke, The partial clone of linear terms, *Siberian Math. J.* **57**(4) (2016) 589–598.
4. K. Denecke and P. Jampachon, Regular elements and Green's relations in menger algebras, *Discuss. Math. Gen. Algebra Appl.* **26** (2006) 85–109.
5. K. Denecke and S. Leeratanavalee, Kernels of generalized hypersubstitutions, in *Proc. of the Sixth Int. Conf.* South-West University, Blagoevgrad, Bulgaria, 2001, August 31–September 2, pp. 87–96.
6. K. Denecke and S. L. Wismath, *Universal Algebra and Applications in Theoretical Computer Sciences* (Chapman & Hall, London, 2002).
7. J. A. Green, On the structure of semigroups, *Ann. Math.* **54**(1) (1951) 163–172.
8. L. Lohapan and P. Jampachon, Semigroup properties of linear terms, *Asian-European J. Math.* **10**(1) (2017), Article ID: 1750051, 11 pp.