แบบฟอร์มแจ้งความประสงค์การใช้งบประมาณสำหรับการพัฒนาบุคลากรคณะวิทยาศาสตร์

ประจำปีงบประมาณ พ.ศ. ๒๕๖๒

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ข้าพเจ้า นางสินดนา คุมวงษ์ ตำแหน่ง ผศ. สังกัด สาขาวิชาคณิตศาสตร์

ได้ขออนุญาตเข้าร่วม ประชุมวิชาการระดับนานาชาติ ICMSA 2018

ตาม**<u>หนังสือขออนุญาต</u>** ศธ.๐๕๒๓.๔.๕ / 405 ลงวันที่ 22 พ.ย. 2561 โดยข้าพเจ้ามีความประสงค์จะขอใช้งบประมาณพัฒนาบุคลากรของคณะวิทยาศาสตร์เพื่อไปพัฒนาตนเอง ดังนี้

☐ <u>กรณีที่ ๑</u> ใช้งบประมาณไม่เกิน ๖,๐๐๐ บาท สำหรับการเข้าร่วมอบรม สัมมนา หรือประชุมวิชาการทั่วไปที่เกี่ยวกับการพัฒนาวิชาชีพของตนเองฯ (ไม่ต้องรายงาน)

☐ <u>กรณีที่ ๒</u> ใช้งบประมาณไม่เกิน ๘,๐๐๐ บาท สำหรับการเข้าร่วมอบรม ฝึกอบรม สัมมนา หรือประชุมวิชาการทั่วไปที่เกี่ยวกับการพัฒนาวิชาชีพของตนเอง ต้องส่งรายงานสรุปเนื้อหาและการนำไปใช้ประโยชน์ อย่างน้อย ๑ หน้ากระดาษ A๔ (เนื้อหาสรุปไม่น้อยกว่า ๒๕ บรรทัด)

☑ <u>กรณีที่ ๓</u> สำหรับการเข้าร่วมนำเสนอผลงานวิชาการในรูปแบบโปสเตอร์ หรือปากเปล่า โดยต้องเป็นผู้เขียนชื่อแรก (First author) หรือต้องเป็นผู้เขียนหลัก (Corresponding author) ซึ่งได้รับการตอบรับเป็นที่เรียบร้อยแล้ว

- คนละไม่เกิน ๑๕,๐๐๐ บาท (สำหรับสายวิชาการ)
- คนละไม่เกิน ๑๐,๐๐๐ บาท (สำหรับสายสนับสนุนวิชาการ)

<u>โดยต้องจัดส่งเอกสาร ดังนี้</u> สำเนาบทคัดย่อ หรือโปสเตอร์(ย่อขนาด A๔) หรือบทความฯ ฉบับเต็ม *<u>และ</u>*ต้องทำรายงานสรุปเนื้อหาและการนำไปใช้ประโยชน์ของการเข้าอบรม อย่างน้อย ๑ หน้ากระดาษ A๔ (เนื้อหาสรุปไม่น้อยกว่า ๒๕ บรรทัด)

☐ <u>กรณีที่ ๔</u> สำหรับการเข้าร่วมอบรมเชิงปฏิบัติการเพื่อเพิ่มสมรรถนะในสายวิชาชีพที่เชี่ยวชาญตามตำแหน่งงานของตนเอง

- คนละไม่เกิน ๑๕,๐๐๐ บาท (สำหรับสายวิชาการ)
- คนละไม่เกิน ๑๐,๐๐๐ บาท (สำหรับสายสนับสนุนวิชาการ)

<u>โดยต้องจัดส่งเอกสาร ดังนี้</u> สำเนาใบรับรองหรือหนังสือรับรองหรือใบประกาศนียบัตรหรือวุฒิบัตร จากการเข้าอบรมเชิงปฏิบัติการ *<u>และ</u>*รายงานสรุปเนื้อหาและการนำไปใช้ประโยชน์ อย่างน้อย ๑ หน้ากระดาษ A๔ (เนื้อหาสรุปไม่น้อยกว่า ๒๕ บรรทัด)

ในปีงบประมาณ พ.ศ. ๒๕๖๑ (๑ ต.ค. ๒๕๖๑ –๓๐ ก.ย ๒๕๖๒) ข้าพเจ้าได้ใช้งบพัฒนาบุคลากรฯ ไปแล้ว จำนวนทั้งสิ้น .... ครั้ง ดังต่อไปนี้

–ครั้งที่ .... ในกรณีที่............ ใช้งบประมาณไปแล้วเป็นจำนวนเงินทั้งสิ้น...บาท

–ครั้งที่ .... ในกรณีที่............ ใช้งบประมาณไปแล้วเป็นจำนวนเงินทั้งสิ้น...บาท

(หากมีจำนวนครั้งเกินกว่านี้ ให้ทำรายละเอียดแนบท้ายเพิ่มเติม)

.... ผู้ขออนุญาต

(นางสินดนา คุมวงษ์)

21 / พ.ย. / 61

.... ประธานหลักสูตร/เลขานุการคณะ/หัวหน้างาน

(นางสินดนา คุมวงษ์)

21 / พ.ย. / 61.

หมายเหตุ : ๑. งบประมาณที่ใช้สำหรับการพัฒนาบุคลากร หมายรวมถึงค่าใช้จ่ายทุกประเภทที่ใช้ในการเข้าร่วมการอบรม/สัมมนา/ประชุม เช่น ค่าลงทะเบียน ค่าใช้จ่ายในการเดินทาง และอื่น ๆ ที่เกี่ยวข้อง

๒. การใช้งบประมาณพัฒนาบุคลากรในที่คณะวิทยาศาสตร์จัดสรร ให้ถือปฏิบัติตามเงื่อนไขที่ได้กำหนดไว้ในแต่ละกรณี

๓. ให้แนบแบบฟอร์มแจ้งความประสงค์ฯ นี้มาพร้อมการส่งรายงานสรุปเนื้อหาและการนำไปใช้ประโยชน์ฯ ด้วย

เห็นชอบตามมติที่ประชุมคณะกรรมการประจำคณะฯ ครั้งที่1/2560

เริ่มใช้ตั้งแต่เดือน 1 กุมภาพันธ์ 2560

รายงานสรุปเนื้อหาและการนำไปใช้ประโยชน์จากการเข้าอบรม สัมมนา หรือประชุมวิชาการ

ข้าพเจ้านางจินตนา จูมวงษ์ ตำแหน่ง ผู้ช่วยศาสตราจารย์ สังกัด หลักสูตร วท.บ. สาขาวิชาคณิตศาสตร์ คณะวิทยาศาสตร์ ขอนำเสนอรายงานสรุปเนื้อหาและการนำไปใช้ประโยชน์จากการเข้าร่วมโครงการ " ประชุมวิชาการระดับนานาชาติ : The 14th IMT-GT International Conference on Mathematics, Statistics and their Applications (ICMSA 2018)" เมื่อวันที่ 9-10 ธันวาคม 2561 ณ โรงแรมสยามออเรียนทัล อำเภอหาดใหญ่ จังหวัดสงขลา ซึ่งจัดโดย คณะวิทยาศาสตร์ มหาวิทยาลัยทักษิณ ตามหนังสือขออนุมัติเดินทางไปปฏิบัติงาน เลขที่ ศธ.0523.4. 5/ 410 ลงวันที่ 28 พฤศจิกายน 2561 ซึ่งการเข้าร่วมโครงการดังกล่าว ข้าพเจ้าได้เลือกใช้งบประมาณการพัฒนาบุคลากร <u>ตามกรณีที่ 3</u>

ดังนั้นจึงขอนำเสนอสรุปเนื้อหาและการนำไปใช้ประโยชน์ของการสัมมนา ดังต่อไปนี้

โครงการ " ประชุมวิชาการระดับนานาชาติ : The 14th IMT-GT International Conference on Mathematics, Statistics and their Applications (ICMSA 2018)" เมื่อวันที่ 9-10 ธันวาคม 2561 จัดโดย คณะวิทยาศาสตร์ มหาวิทยาลัยทักษิณ

สรุปเนื้อหา

1. เป็นกิจกรรมที่จัดขึ้นเพื่อเผยแพร่ผลงานทางวิชาการและงานวิจัย ทางด้าน คณิตศาสตร์ สถิติ และการประยุกต์ อีกทั้งเป็นการสร้างเครือข่ายระหว่างนักวิจัยทั้งภายในและต่างประเทศ แลกเปลี่ยนความรู้และประสบการณ์ในการทำวิจัยร่วมกัน
ซึ่งแบ่งการนำเสนองาน ดังนี้
    - Oral Presentation sessions 1
    - Oral Presentation sessions 2
    - Oral Presentation sessions 3
    - Oral Presentation sessions 4
    - Poster Presentation
2. รับฟังการบรรยายพิเศษ จาก invited speaker ดังนี้
    1) Prof. Dr. Suthep Suantai "Inertial S-Iteration FORWARD-BACKWARD Algorithm for a Family of Nonexpansive Operators with Applications to Regression and Image Restoration Problems " from THAILAND.
    2) Prof. Dr. Dedi Rosadi "Automatic Forecasting Method of the Term Structure of Government Bond Yields" from INDONESIA.

3) Prof. Dr. Habshah Midi "Robust Statistics: Handling of Outliers for Efficient Prediction" from MALAYSIA.

4) Asst. Dr. Winai Bodhisuwan "Statistical Analysis of Count Data" from THAILAND.

3. ข้าพเจ้าได้นำเสนอผลงานวิจัยในรูปแบบบรรยาย (เอกสารแนบ บทความฉบับเต็ม) เรื่อง **Linear-Hypersubstitutions for Algebraic Systems of type ((n);(n)) and Characterization of their Idempotent elements**

Session Information

Date : 9th December 2018
Time : 03:15-05:00 p.m.
Room : 1
Title : Mathematics
Chair : Prof. Dr. Suthep Suantai

Time	Paper ID	Authors	Paper Title	Speaker
03:15-03:30 p.m.	55	Jintana Joomwong and Dara Phusanga	Linear-Hypersubstitutions for Algebraic Systems of type ((n);(n)) and Characterization of their idempotent elements	Jintana Joomwong

4. ได้รับรางวัลในการนำเสนอผลงานวิจัยในรูปแบบบรรยาย

The Honorable Mention Presentation in Mathematics and Applied Mathematics


ICMSA 2018 IMT-GT

CERTIFICATE OF APPRECIATION

Awarded to

Jintana Joomwong

For The Honorable Mention Presentation in Mathematics and Applied Mathematics

Entitled

Linear Hypersubstitutions for Algebraic Systems of type ((n);(n)) and Characterization of their idempotent elements

in

14th International Conference on Mathematics, Statistics and Their Applications (ICMSA 2018)

held on December 8-10, 2018 in Thaksin University, Thailand

President of Thaksin University

Organized by

Thaksin University

การนำไปใช้ประโยชน์

จากการเข้าร่วมกิจกรรมในครั้งนี้ ได้พบปะและแลกเปลี่ยนแนวคิดทำให้มองเห็นแนวทางการทำงานวิจัยในอนาคต อีกทั้งเห็นแนวทางในการพัฒนาการเรียนการสอนในหลักสูตรที่รับผิดชอบ และแนวทางการสร้างความร่วมมือทางวิชาการจากบุคคลากรของสถาบันการศึกษาที่เข้าร่วมประชุม ซึ่งเป็นประโยชน์โดยตรงกับสายงานของผู้รายงาน

....

(นางจินตนา จูมวงษ์)

2 มกราคม 2562

ความคิดเห็นของผู้บังคับบัญชาชั้นต้น (ประธานหลักสูตร/เลขานุการคณะ/หัวหน้างาน)

เป็นกิจกรรมที่สามารถพัฒนาศักยภาพของผู้รายงาน

และพัฒนาหลักสูตรต่อไปได้ในอนาคต

....

(ผู้ช่วยศาสตราจารย์จินตนา จูมวงษ์)

ประธานหลักสูตร วท.บ. สาขาวิชาคณิตศาสตร์

ความคิดเห็นของคณบดีคณะวิทยาศาสตร์หรือผู้แทน

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ICMSA 2018

IMT-GT

Indonesia - Malaysia - Thailand
Growth Triangle

CERTIFICATE OF APPRECIATION

Awarded to

Jintana Joomwong

For The Honorable Mention Presentation in Mathematics and Applied Mathematics

Entitled

Linear-Hypersubstitutions for Algebraic Systems of type ((n);(n)) and Characterization of their Idempotent elements

in

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Associate Professor Dr. Wichai Chumni

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Linear-Hypersubstitutions for Algebraic Systems of type $((n);(n))$ and Characterization of their Idempotent elements

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Abstract : A formula in which each variable occurs at most once is said to be a linear-formula ([5], [6]). A linear-hypersubstitution for algebraic systems of type $((n);(n))$ is a mapping $\sigma_{\iota,F}$ which maps n-ary operation symbols f to n-ary linear-terms $\sigma_{\iota,F}(f)$ and n-ary relational symbols γ to n-ary linear-formulas $\sigma_{\iota,F}(\gamma)$. Any linear-hypersubstitution $\sigma_{\iota,F}$ can be extended to a mapping $\widehat{\sigma}_{\iota,F}$ on the set of all linear-terms of type (n) and linear-formulas of type $((n);(n))$. A binary operation "$\circ_{lin}$" on $Hyp^{lin}((n);(n))$ the set of all linear-hypersubstitutions for algebraic systems of type $((n);(n))$ can be defined by using this extension. The set $Hyp^{lin}((n);(n))$ together with the identity linear-hypersubstitution $(\sigma_{\iota,F})_{id}$ which maps $(\sigma_{\iota,F})_{id}(f) := f(x_1,\ldots,x_n)$ and $(\sigma_{\iota,F})_{id}(\gamma) := \gamma(x_1,\ldots,x_n)$ forms a monoid. The concept of an idempotent element plays an important role in semigroup theory [3]. In this paper, we characterize the idempotent of $(Hyp^{lin}((n);(n));\circ_{lin},(\sigma_{\iota,F})_{id})$.

Keywords : algebraic system, formula , linear-hypersubstitution, idempotent.
2010 AMS Mathematics classification : 20M07

1 Introduction

Algebraic systems are understood in the sence of Mal'cev(see [1]). An *algebraic system* of type (τ,τ') is a triple $\mathcal{A} := (A;(f_i^{\mathcal{A}})_{i\in I},(\gamma_j^{\mathcal{A}})_{j\in J})$ consisting of a non-empty set A, an indexed set $(f_i^{\mathcal{A}})_{i\in I}$ of operations defined on A where $f_i^{\mathcal{A}} : A^{n_i}\to A$ is n_i-ary and an indexed set of relations $\gamma_j^{\mathcal{A}}\subseteq A^{n_j}$ is an n_j-ary . The pair (τ,τ') with $\tau=(n_i)_{i\in I}$, $\tau'=(n_j)_{j\in J}$ of sequences of positive integers n_i,n_j is called the *type* of $\mathcal{A}$.

The concept of a term and a formula are one of the fundamental concepts of algebraic system. To be independent, first we repeat the most important definitions and results on hypersubstitutions for algebraic systems (see [2]). Using for $n\geq 1$, an n-ary alphabet $X_n=\{x_1,x_2,\ldots,x_n\}$ of individual variables and the alphabet $(f_i)_{i\in I}$ of operation symbols in the usual way one defines terms of type τ by the following steps :

(i) Every $x_l\in X_n$ is an n-ary term of type τ.

(ii) If $t_1,\ldots,t_{n_i}$ are n-ary terms of type τ and if f_i is an n_i-ary operation symbol of type τ, then $f_i(t_1,\ldots,t_{n_i})$ is an n-ary term of type τ.

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Let $W_\tau(X_n)$ be the set of all n-ary terms of type τ. If $X = \{x_1, x_2, \ldots\}$ is a countably infinite alphabet, then $W_\tau(X) := \bigcup_{n\geq 1} W_\tau(X_n)$ denote the set of all terms of type τ (see [7]).

To define quantifier free formulas of type (τ, τ'), we need the logical connectives $\neg$ (for negation), $\vee$ (for disjunction) and the equation symbol $\approx$.

Definition 1.1. *Let $n \in \mathbb{N}$. An n-ary quantifier free formula of type (τ, τ') (for short, formula of type (τ, τ')) is defined in the following inductive way :*

(i) *If t_1, t_2 are n-ary terms of type τ, then the equation $t_1 \approx t_2$ is an n-ary quantifier free formula of type (τ, τ').*

(ii) *If $j \in J$ and $t_1, \ldots, t_{n_j}$ are n-ary terms of type τ, then $\gamma_j(t_1, \ldots, t_{n_j})$ is an n-ary quantifier free formula of type (τ, τ').*

(iii) *If F is an n-ary quantifier free formula of type (τ, τ'), then $\neg F$ is an n-ary quantifier free formula of type (τ, τ').*

(iv) *If F_1 and F_2 are n-ary quantifier free formulas of type (τ, τ'), then $F_1 \vee F_2$ is an n-ary quantifier free formula of type (τ, τ').*

Let $\mathcal{F}_{(\tau,\tau')}(X_n)$ be the set of all n-ary quantifier free formulas of type (τ, τ') and let $\mathcal{F}_{(\tau,\tau')}(X)$ $:= \bigcup_{n\geq 1} \mathcal{F}_{(\tau,\tau')}(X_n)$ be the set of all quantifier free formulas of type (τ, τ').

2 Linear-terms of type τ and Linear-formulas of type (τ, τ')

A term in which each variable occurs at most once, is said to be a linear. For a formal definition of n-ary-linear-terms, we replace (ii) in the definition of terms by a slightly different condition. Let $var(t)$ is the set of all variables occuring in a term t and $var(F)$ is the set of all variables occuring in a formula F.

Definition 2.1. *Let $n \in \mathbb{N}$. An n-ary linear-term of type τ is defined in the following inductive way :*

(i) *Every $x_j \in X_n$ is an n-ary linear-term of type τ.*

(ii) *If $t_1, \ldots, t_{n_i}$ are n-ary linear-terms of type τ and if $var(t_l) \cap var(t_k) = \emptyset$ for all $1 \leq l < k \leq n_i$, then $f_i(t_1, \ldots, t_{n_i})$ is an n-ary linear-term of type τ.*

(iii) *The set $W_\tau^{lin}(X_n)$ of all n-ary linear-terms of type τ is the smallest set which contains $x_1, \ldots, x_n$ and closed under finite applications of (ii)*

The set of all linear-terms of type τ over the countably infinite alphabet X is defined by $W_\tau^{lin}(X) := \bigcup_{n\geq 1} W_\tau^{lin}(X_n)$.

Definition 2.2. *Let $n \in \mathbb{N}$. An n-ary linear-formula of type (τ, τ') is defined by the following inductive way :*

(i) *If t_1, t_2 are n-ary linear-terms of type τ and $var(t_1) \cap var(t_2) = \emptyset$, then the equation $t_1 \approx t_2$ is an n-ary linear-formula of type (τ, τ').*

(ii) *If $t_1, \ldots, t_{n_j}$ are n-ary linear-terms of type τ, $var(t_l) \cap var(t_k) = \emptyset$; $l, k \in \{1, 2, \ldots, n_j\}$ and γ_j is an n_j-ary relational symbol, then $\gamma_j(t_1, \ldots, t_{n_j})$ is an n-ary linear-formula of type (τ, τ').*

(iii) *If F is an n-ary linear-formula of type (τ, τ'), then $\neg F$ is an n-ary linear-formula of type (τ, τ').*

(iv) *If F_1 , F_2 are n-ary linear-formulas of type (τ, τ') and $var(F_1) \cap var(F_2) = \emptyset$, then $F_1 \vee F_2$ is an n-ary linear-formula of type (τ, τ').*

Let $\mathcal{F}^{lin}_{(\tau,\tau')}(X_n)$ be the set of all n-ary linear-formulas of type (τ,τ') and let $\mathcal{F}^{lin}_{(\tau,\tau')}(X) := \bigcup_{n\geq 1} \mathcal{F}^{lin}_{(\tau,\tau')}(X_n)$ be the set of all linear-formulas of type (τ,τ').

For this paper, we consider the type $(\tau,\tau') := ((n);(n))$, then $f(t_1,\ldots,t_n)$ can not be a linear-term, where $t_1,\ldots,t_n \in W_n(X_n)\setminus X_n$ and $F_1 \vee F_2$ can not be a linear-formula, because $var(F_1)\cap vae(F_2) \neq \emptyset$ as the following an Example 2.3.

Example 2.3. *: Let $(\tau,\tau') := ((2),(2))$ with a binary operation symbol f and a binary relational symbol γ and let $X_2 = \{x_1,x_2\}$. Then $W^{lin}_{(2)}(X_2) = \{x_1,x_2,f(x_1,x_2),f(x_2,x_1)\}$ and $\mathcal{F}^{lin}_{((2),(2))}(X_2) = \{x_1 \approx x_2, x_2 \approx x_1, \gamma(x_1,x_2), \gamma(x_2,x_1), \neg(x_1\approx x_2), \neg(x_2 \approx x_1), \neg(\gamma(x_1,x_2)), \neg(\gamma(x_2,x_1)), \neg(\neg(x_1\approx x_2)),\ldots\}$.*

3 Superposition of Linear-Terms and Linear-Formulas of type $((n);(n))$

Substituting the variables occuring in a linear-term by other linear-terms one obtains a new linear-term. This can be described by the superposition operation $S^n_{lin}, n \geq 1$ for linear-terms which is inductively defined as follows :

Definition 3.1. *Let $n \in \mathbb{N}$ and $t,t_1,\ldots,t_n \in W^{lin}_n(X_n)$ such that $var(t_l)\cap var(t_k) = \emptyset$, for $l,k \in \{1,\ldots,n\}$. The operation*

$$S^n_{lin} : W^{lin}_n(X_n) \times (W^{lin}_n(X_n))^n \to W^{lin}_n(X_n)$$

is defined in the following inductive way :

(i) *If $t = x_i$, then $S^n_{lin}(x_i,t_1,\ldots,t_n) := t_i$; $1 \leq i \leq n$,*

(ii) *If $t = f(s_1,\ldots,s_n)$ and assume that, $S^n_{lin}(s_l,t_1,\ldots,t_n)$ is a linear-term already, for $l \in \{1,\ldots,n\}$ such that $var(S^n_{lin}(s_l,t_1,\ldots,t_n)) \cap var(S^n_{lin}(s_k,t_1,\ldots,t_n)) = \emptyset$; $1 \leq l,k \leq n$, then $S^n_{lin}(f(s_1,\ldots,s_n),t_1,\ldots,t_n) := f(S^n_{lin}(s_1,t_1,\ldots,t_n),\ldots, S^n_{lin}(s_n,t_1,\ldots,t_n))$.*

Now, we will extend this superposition of linear-terms of type (n) to a superposition of linear-formulas of type $((n),(n))$as follow :

Definition 3.2. *Let $n \in \mathbb{N}$ and $t,t_1,\ldots,t_n \in W^{lin}_n(X_n)$ such that $var(t_l)\cap var(t_k) = \emptyset$; $l,k \in \{1,\ldots,n\}$ and S^n_{lin} be the superposition of linear-terms which have defined above. The operation*

$$R^n_{lin} : W^{lin}_n(X_n) \cup \mathcal{F}^{lin}_{((n);(n))}(X_n) \times (W^{lin}_n(X_n))^n \to W^{lin}_n(X_n) \cup \mathcal{F}^{lin}_{((n);(n))}(X_n)$$

is defined in the following inductive way :

(i) *If $t \in W^{lin}_n(X_n)$, then $R^n_{lin}(t,t_1,\ldots,t_n) := S^n_{lin}(t,t_1,\ldots,t_n)$.*

(ii) *If F has the form $s_1 \approx s_2$ and $var(S^n_{lin}(s_1,t_1,\ldots,t_n))\cap var(S^n_{lin}(s_2,t_1,\ldots,t_n)) = \emptyset$, then $R^n_{lin}(s_1 \approx s_2,t_1,\ldots,t_n) := S^n_{lin}(s_1,t_1,\ldots,t_n) \approx S^n_{lin}(s_2,t_1,\ldots,t_n)$.*

(iii) *If F has the form $\gamma(s_1,\ldots,s_n)$, and assume that, $S^n_{lin}(s_l,t_1,\ldots,t_n)$ is a linear-term already ; $l \in \{1,\ldots,n\}$ such that $var(S^n_{lin}(s_l,t_1,\ldots,t_n)) \cap var(S^n_{lin}(s_k,t_1,\ldots,t_n)) = \emptyset$; $1 \leq l,k \leq n$, then $R^n_{lin}(\gamma(s_1,\ldots,s_n),t_1,\ldots,t_n) := \gamma(S^n_{lin}(s_1,t_1,\ldots,t_n),\ldots,S^n_{lin}(s_n,t_1,\ldots,t_n))$.*

(iv) *If F has the form $\neg F$, and assume that, $R^n_{lin}(F,t_1,\ldots,t_n)$ is a linear-formula already, then $R^n_{lin}(\neg F,t_1,\ldots,t_n) := \neg R^n_{lin}(F,t_1,\ldots,t_n)$.*

Let π be a permutation on the set $\{1,2,\ldots,n\}$.

Theorem 3.3. *Let $\beta \in W_n^{lin}(X_n) \cup \mathcal{F}^{lin}_{((n);(n))}(X_n)$. The operation R^n_{lin} satisfies :*

(LFC1) $R^n_{lin}(R^n_{lin}(\beta, t_1, \ldots, t_n), s_1, \ldots, s_n) = R^n_{lin}(\beta, R^n_{lin}(t_1, s_1, \ldots, s_n), \ldots, R^n_{lin}(t_n, s_1, \ldots, s_n))$ *whenever* $t_1, \ldots, t_n, s_1, \ldots, s_n \in W_n^{lin}(X_n)$ *and* $var(t_l) \cap var(t_k) = \emptyset$, $var(s_l) \cap var(s_k) = \emptyset$; $l, k \in \{1, \ldots, n\}$.

(LFC2) $R^n_{lin}(x_i, t_1, \ldots, t_n) = t_i$ *whenever* $t_1, \ldots, t_n \in W_n^{lin}(X_n)$ *and* $var(t_l) \cap var(t_k) = \emptyset$; $l, k \in \{1, \ldots, n\}$.

(LFC3) $R^n_{lin}(\beta, x_1, \ldots, x_n) = \beta$.

Proof. For $\beta = t \in W_n^{lin}(X_n)$, we will give a proof of (LFC1) by induction on the complexity of a linear-term t.

(i) If $t = x_i$; $1 \leq i \leq n$, then

$$\begin{aligned} & R^n_{lin}(R^n_{lin}(x_i, t_1, \ldots, t_n), s_1, \ldots, s_n) \\ = \;& R^n_{lin}(S^n_{lin}(x_i, t_1, \ldots, t_n), s_1, \ldots, s_n) \\ = \;& S^n_{lin}(t_i, s_1, \ldots, s_n) \\ = \;& S^n_{lin}(x_i, S^n_{lin}(t_1, s_1, \ldots, s_n), \ldots, S^n_{lin}(t_n, s_1, \ldots, s_n)) \\ = \;& R^n_{lin}(x_i, R^n_{lin}(t_1, s_1, \ldots, s_n), \ldots, R^n_{lin}(t_n, s_1, \ldots, s_n)). \end{aligned}$$

(ii) If $t = f(x_{\pi(1)}, \ldots, x_{\pi(n)})$ and assume that $S^n_{lin}(S^n_{lin}(x_{\pi(l)}, t_1, \ldots, t_n), s_1, \ldots, s_n) = S^n_{lin}(x_{\pi(l)}, S^n_{lin}(t_1, s_1, \ldots, s_n), \ldots, S^n_{lin}(t_n, s_1, \ldots, s_n))$; $1 \leq l \leq n$, then

$$\begin{aligned} & R^n_{lin}(R^n_{lin}(f(x_{\pi(1)}, \ldots, x_{\pi(n)}), t_1, \ldots, t_n), s_1, \ldots, s_n) \\ = \;& R^n_{lin}(S^n_{lin}(f(x_{\pi(1)}, \ldots, x_{\pi(n)}), t_1, \ldots, t_n), s_1, \ldots, s_n) \\ = \;& R^n_{lin}(f(S^n_{lin}(x_{\pi(1)}, t_1, \ldots, t_n), \ldots, S^n_{lin}(x_{\pi(n)}, t_1, \ldots, t_n)), s_1, \ldots, s_n) \\ = \;& f(S^n_{lin}(S^n_{lin}(x_{\pi(1)}, t_1, \ldots, t_n), s_1, \ldots, s_n), \ldots, S^n_{lin}(S^n_{lin}(x_{\pi(n)}, t_1, \ldots, t_n), s_1, \ldots, s_n)) \\ = \;& f(S^n_{lin}(x_{\pi(1)}, S^n_{lin}(t_1, s_1, \ldots, s_n), \ldots, S^n_{lin}(t_n, s_1, \ldots, s_n)), \ldots, \\ & S^n_{lin}(x_{\pi(n)}, S^n_{lin}(t_1, s_1, \ldots, s_n), \ldots, S^n_{lin}(t_n, s_1, \ldots, s_n))) \\ = \;& S^n_{lin}(f(x_{\pi(1)}, \ldots, x_{\pi(n)}), S^n_{lin}(t_1, s_1, \ldots, s_n), \ldots, S^n_{lin}(t_n, s_1, \ldots, s_n)) \\ = \;& R^n_{lin}(f(x_{\pi(1)}, \ldots, x_{\pi(n)}), R^n_{lin}(t_1, s_1, \ldots, s_n), \ldots, R^n_{lin}(t_n, s_1, \ldots, s_n)). \end{aligned}$$

For $\beta = F \in \mathcal{F}^{lin}_{((n);(n))}(X_n)$, we will give a proof of (LFC1) by induction on the complexity of a linear-formula F.

(i) If F has the form $x_i \approx x_j; i \neq j \in \{1, \ldots, n\}$, then

$$\begin{aligned} & R^n_{lin}(R^n_{lin}(x_i \approx x_j, t_1, \ldots, t_n), s_1, \ldots, s_n) \\ = \;& R^n_{lin}(S^n_{lin}(x_i, t_1, \ldots, t_n) \approx S^n_{lin}(x_j, t_1, \ldots, t_n), s_1, \ldots, s_n) \\ = \;& S^n_{lin}(S^n_{lin}(x_i, t_1, \ldots, t_n), s_1, \ldots, s_n) \approx S^n_{lin}(S^n_{lin}(x_j, t_1, \ldots, t_n), s_1, \ldots, s_n) \\ = \;& S^n_{lin}(x_i, S^n_{lin}(t_1, s_1, \ldots, s_n), \ldots, S^n_{lin}(t_n, s_1, \ldots, s_n)) \approx \\ & S^n_{lin}(x_j, S^n_{lin}(t_1, s_1, \ldots, s_n), \ldots, S^n_{lin}(t_n, s_1, \ldots, s_n)) \\ = \;& R^n_{lin}(x_i \approx x_j, R^n_{lin}(t_1, s_1, \ldots, s_n), \ldots, R^n_{lin}(t_n, s_1, \ldots, s_n)). \end{aligned}$$

(ii) If F has the form $\gamma(x_{\pi(1)}, \ldots, x_{\pi(n)})$, then

$$\begin{aligned} & R^n_{lin}(R^n_{lin}(\gamma(x_{\pi(1)}, \ldots, x_{\pi(n)}), t_1, \ldots, t_n), s_1, \ldots, s_n) \\ = \;& R^n_{lin}(\gamma(S^n_{lin}(x_{\pi(1)}, t_1, \ldots, t_n), \ldots, S^n_{lin}(x_{\pi(n)}, t_1, \ldots, t_n)), s_1, \ldots, s_n) \\ = \;& \gamma(S^n_{lin}(S^n_{lin}(x_{\pi(1)}, t_1, \ldots, t_n), s_1, \ldots, s_n), \ldots, S^n_{lin}(S^n_{lin}(x_{\pi(n)}, t_1, \ldots, t_n), s_1, \ldots, s_n)) \\ = \;& \gamma(S^n_{lin}(x_{\pi(1)}, S^n_{lin}(t_1, s_1, \ldots, s_n), \ldots, S^n_{lin}(t_n, s_1, \ldots, s_n)), \ldots, \\ & S^n_{lin}(x_{\pi(n)}, S^n_{lin}(t_1, s_1, \ldots, s_n), \ldots, S^n_{lin}(t_n, s_1, \ldots, s_n))) \\ = \;& R^n_{lin}(\gamma(x_{\pi(1)}, \ldots, x_{\pi(n)}), R^n_{lin}(t_1, s_1, \ldots, s_n), \ldots, R^n_{lin}(t_n, s_1, \ldots, s_n)). \end{aligned}$$

(iii) If F has the form $\neg F$ and assume that $R^n_{lin}(R^n_{lin}(F, t_1, \ldots, t_n), s_1, \ldots, s_n) = R^n_{lin}(F, R^n_{lin}(t_1, s_1, \ldots, s_n), \ldots, R^n_{lin}(t_n, s_1, \ldots, s_n))$, then

$$\begin{aligned} & R^n_{lin}(R^n_{lin}(\neg F, t_1, \ldots, t_n), s_1, \ldots, s_n) \\ = \;& R^n_{lin}(\neg R^n_{lin}(F, t_1, \ldots, t_n), s_1, \ldots, s_n) \\ = \;& \neg R^n_{lin}(R^n_{lin}(F, t_1, \ldots, t_n), s_1, \ldots, s_n) \\ = \;& R^n_{lin}(\neg F, R^n_{lin}(t_1, s_1, \ldots, s_n), \ldots, R^n_{lin}(t_n, s_1, \ldots, s_n)). \end{aligned}$$

(LFC2) is clearly by Definition 3.1(i).
The proof of (LFC3), we will proceed in a similar way considering the completely of a linear-term t.

(i) If $t = x_i$; $1 \leq i \leq n$, then
$R^n_{lin}(x_i, x_1, \ldots, x_n) = S^n_{lin}(x_i, x_1, \ldots, x_n) = x_i$.

(ii) If $t = f(x_{\pi(1)}, \ldots, x_{\pi(n)})$ and assume that $R^n_{lin}(x_{\pi(l)}, x_1, \ldots, x_n) = x_{\pi(l)}$; $1 \leq l \leq n$, then
$$\begin{aligned} &R^n_{lin}(f(x_{\pi(1)}, \ldots, x_{\pi(n)}), x_1, \ldots, x_n) \\ &= S^n_{lin}(f(x_{\pi(1)}, \ldots, x_{\pi(n)}), x_1, \ldots, x_n) \\ &= f(S^n_{lin}(x_{\pi(1)}, x_1 \ldots, x_n), \ldots, S^n_{lin}(x_{\pi(n)}, x_1 \ldots, x_n)) \\ &= f(x_{\pi(1)}, \ldots, x_{\pi(n)}). \end{aligned}$$

Next, we will proceed in a similar way considering the completely of linear-formula F.

(i) If F has the form $x_i \approx x_j, i \neq j \in \{1, \ldots, n\}$, then
$R^n_{lin}(x_i \approx x_j, x_1, \ldots, x_n) = S^n_{lin}(x_i, x_1, \ldots, x_n) \approx S^n_{lin}(x_j, x_1, \ldots, x_n) = x_i \approx x_j$.

(ii) If F has the form $\gamma(x_{\pi(1)}, \ldots, x_{\pi(n)})$, then
$$\begin{aligned} &R^n_{lin}(\gamma(x_{\pi(1)}, \ldots, x_{\pi(n)}), x_1, \ldots, x_n) \\ &= \gamma(S^n_{lin}(x_{\pi(1)}, x_1 \ldots, x_n), \ldots, S^n_{lin}(x_{\pi(n)}, x_1 \ldots, x_n)) = \gamma(x_{\pi(1)}, \ldots, x_{\pi(n)}). \end{aligned}$$

(iii) If F has the form $\neg F$ and assume that $R^n_{lin}(F, x_1, \ldots, x_n) = F$, then
$R^n_{lin}(\neg F, x_1, \ldots, x_n) = \neg R^n_{lin}(F, x_1, \ldots, x_n) = \neg F$.

□

4 Linear-Hypersubstitutions for Algebraic Systems of type ((n);(n))

The concept of linear-hypersubstitutions for algebra was introduced by Th.Changphas, K.Denecke and B.P.baljomme [9]. We are going to extend this concept to algebraic system of type $((n);(n))$ as the following:

Definition 4.1. *Any mapping*

$$\sigma : \{f\} \cup \{\gamma\} \to W^{lin}_n(X_n) \cup \mathcal{F}^{lin}_{((n);(n))}(X_n)$$

which maps operation symbols f to linear-terms and relational symbols γ to linear-formulas preserving arities is called a linear-hypersubstitution for algebraic systems (of type $((n);(n))$).

Let $Hyp^{lin}((n);(n))$ be the set of all linear-hypersubstitutions for algebraic systems of type $((n);(n))$.

We can define an extension of linear-hypersubstitutions for algebraic systems of type $((n);(n))$ as follows:

$$\widehat{\sigma} : W^{lin}_n(X_n) \cup \mathcal{F}^{lin}_{((n);(n))}(X_n) \to W^{lin}_n(X_n) \cup \mathcal{F}^{lin}_{((n);(n))}(X_n)$$

(i) $\widehat{\sigma}[x] := x$ for any variable $x \in X_n$,

(ii) $\widehat{\sigma}[f(x_{\pi(1)}, \ldots, x_{\pi(n)})] := S^n_{lin}(\sigma(f), \widehat{\sigma}[x_{\pi(1)}], \ldots, \widehat{\sigma}[x_{\pi(n)}])$,

(iii) $\widehat{\sigma}[x_i \approx x_j] := \widehat{\sigma}[x_i] \approx \widehat{\sigma}[x_j]$ for $i \neq j \in \{1, \ldots, n\}$,

(iv) $\widehat{\sigma}[\gamma(x_{\pi(1)}, \ldots, x_{\pi(n)})] := R^n_{lin}(\sigma(\gamma), \widehat{\sigma}[x_{\pi(1)}], \ldots, \widehat{\sigma}[x_{\pi(n)}])$,

(v) $\widehat{\sigma}[\neg F] := \neg\widehat{\sigma}[F]$ for $F \in \mathcal{F}^{lin}_{((n);(n))}(X_n)$.

Then $\widehat{\sigma}$ is called the extension of linear-hypersubstitution σ for algebraic system.

Next, we defined a binary operation $\circ_{lin}$ on $Hyp^{lin}((n);(n))$ by $\sigma_1 \circ_{lin} \sigma_2 := \widehat{\sigma}_1 \circ \sigma_2$ where $\circ$ denotes the usual composition of mapping and $\sigma_1, \sigma_2 \in Hyp^{lin}((n);(n))$. The purpose of this paper, the structure $(Hyp^{lin}((n);(n)), \circ_{lin}, \sigma_{id})$ becomes a monoid. We have to use many tools to prove that.

Lemma 4.2. *For $n \in \mathbb{N}$, let $\sigma \in Hyp^{lin}((n);(n))$, and let $t_1, ..., t_n \in W_n^{lin}(X_n)$ and $var(t_l) \cap var(t_k) = \emptyset; 1 \leq l, k \leq n$. Then*

$$\widehat{\sigma}[R_{lin}^n(\beta, t_1, ..., t_n)] = R_{lin}^n(\widehat{\sigma}[\beta], \widehat{\sigma}[t_1], ..., \widehat{\sigma}[t_n])$$

for any $\beta \in W_n^{lin}(X_n) \cup \mathcal{F}_{((n);(n))}^{lin}(X_n)$.

Proof. For $\beta = t \in W_n^{lin}(X_n)$, we will give a proof by induction on the complexity of the definition of a linear-term t as follows :

(i) If $t = x_i; 1 \leq i \leq n$, then
$\widehat{\sigma}[S_{lin}^n(x_i, t_1, ..., t_n)] = \widehat{\sigma}[t_i] = S_{lin}^n(x_i, \widehat{\sigma}[t_1], ..., \widehat{\sigma}[t_n]) = S_{lin}^n(\widehat{\sigma}[x_i], \widehat{\sigma}[t_1], ..., \widehat{\sigma}[t_n])$.

(ii) $t = f(x_{\pi(1)}, \ldots, x_{\pi(n)})$, and assume that
$\widehat{\sigma}\left[S_{lin}^n\left(x_{\pi(l)}, t_1, ..., t_n\right)\right] = S_{lin}^n(\widehat{\sigma}[x_{\pi(l)}], \widehat{\sigma}[t_1], ..., \widehat{\sigma}[t_n]);\ 1 \leq l \leq n$, then

$$\begin{aligned}
&\widehat{\sigma}\left[S_{lin}^n\left(f(x_{\pi(1)}, \ldots, x_{\pi(n)}), t_1, ..., t_n\right)\right] \\
&= \widehat{\sigma}[f(S_{lin}^n(x_{\pi(1)}, t_1, \ldots, t_n), \ldots, S_{lin}^n(x_{\pi(n)}, t_1, \ldots, t_n))] \\
&= S_{lin}^n(\sigma(f), \widehat{\sigma}[S_{lin}^n(x_{\pi(1)}, t_1, \ldots, t_n)], \ldots, \widehat{\sigma}[S_{lin}^n(x_{\pi(n)}, t_1, \ldots, t_n)]) \\
&= S_{lin}^n(\sigma(f), S_{lin}^n(\widehat{\sigma}[x_{\pi(1)}], \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_n]), \ldots, S_{lin}^n(\widehat{\sigma}[x_{\pi(n)}], \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_n]) \\
&= S_{lin}^n(S_{lin}^n(\sigma(f), \widehat{\sigma}[x_{\pi(1)}], \ldots, \widehat{\sigma}[x_{\pi(n)}]), \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_n]) \\
&= S_{lin}^n(\widehat{\sigma}[f(x_{\pi(1)}, \ldots, x_{\pi(n)})], \widehat{\sigma}[t_1], ..., \widehat{\sigma}[t_n]).
\end{aligned}$$

For $\beta = F \in \mathcal{F}_{((n);(n))}^{lin}(X_n)$, we will give a proof by induction on the complexity of the definition of a linear-formula F as follows :

(i) If F has the form $x_i \approx x_j$ for $i \neq j \in \{1, \ldots, n\}$, then

$$\begin{aligned}
&\widehat{\sigma}[R_{lin}^n(x_i \approx x_j, t_1, \ldots, t_n)] \\
&= \widehat{\sigma}[S_{lin}^n(x_i, t_1, \ldots, t_n) \approx S_{lin}^n(x_j, t_1, \ldots, t_n)] \\
&= S_{lin}^n(\widehat{\sigma}[x_i], \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_n]) \approx S_{lin}^n(\widehat{\sigma}[x_j], \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_n]) \\
&= R_{lin}^n(\widehat{\sigma}[x_i \approx x_j], \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_n]).
\end{aligned}$$

(ii) If F has the form $\gamma(x_{\pi(1)}, \ldots, x_{\pi(n)})$ and assume that
$\widehat{\sigma}[R_{lin}^n(x_{\pi(l)}, t_1, \ldots, t_n)] = R_{lin}^n(\widehat{\sigma}[x_{\pi(l)}], (\widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_n]); 1 \leq l \leq n$, then

$$\begin{aligned}
&\widehat{\sigma}[R_{lin}^n(\gamma(x_{\pi(1)}, \ldots, x_{\pi(n)}), t_1, \ldots, t_n)] \\
&= \widehat{\sigma}[\gamma(S_{lin}^n(x_{\pi(1)}, t_1, \ldots, t_n), \ldots, S_{lin}^n(x_{\pi(n)}, t_1, \ldots, t_n))] \\
&= R_{lin}^n(\sigma(\gamma), \widehat{\sigma}[S_{lin}^n(x_{\pi(1)}, t_1, \ldots, t_n)], \ldots, \widehat{\sigma}[S_{lin}^n(x_{\pi(n)}, t_1, \ldots, t_n)]) \\
&= R_{lin}^n(R_{lin}^n(\sigma(\gamma), \widehat{\sigma}[x_{\pi(1)}], \ldots, \widehat{\sigma}[x_{\pi(n)}]), \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_n]) \\
&= R_{lin}^n(\widehat{\sigma}[\gamma(x_{\pi(1)}, \ldots, x_{\pi(n)})], \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_n]).
\end{aligned}$$

(iii) If F has the form $\neg F$ and assume that
$\widehat{\sigma}[R_{lin}^n(F, t_1, \ldots, t_n)] = R_{lin}^n(\widehat{\sigma}[F], \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_n])$, then

$$\begin{aligned}
&\widehat{\sigma}[R_{lin}^n(\neg F, t_1, \ldots, t_n)] \\
&= \neg(\widehat{\sigma}[R_{lin}^n(F, t_1, \ldots, t_n)]) \\
&= \neg(R_{lin}^n(\widehat{\sigma}[F], \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_n])) \\
&= R_{lin}^n(\widehat{\sigma}[\neg F]), \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_n]).
\end{aligned}$$

□

Lemma 4.3. *For any $\sigma_1, \sigma_2 \in Hyp^{lin}((n);(n))$, we have*

$$(\sigma_1 \circ_{lin} \sigma_2)\widehat{\ } = \widehat{\sigma}_1 \circ \widehat{\sigma}_2.$$

Proof. For $t \in W_n(X_n)$, we will give a proof by induction on the complexity of the definition of a linear-term t.

(i) If $t = x_i$; $1 \le i \le n$,then
$(\sigma_1 \circ_{lin} \sigma_2)^{\frown}[x_i] = x_i = \widehat{\sigma}_1[x_i] = \widehat{\sigma}_1[\widehat{\sigma}_2[x_i]] = (\widehat{\sigma}_1 \circ \widehat{\sigma}_2)[x_i]$.

(ii) If $t = f(x_{\pi(1)}, \ldots, x_{\pi(n)})$, then

$$\begin{aligned}
&(\sigma_1 \circ_{lin} \sigma_2)^{\frown}[f(x_{\pi(1)}, \ldots, x_{\pi(n)})] \\
&= S^n_{lin}((\sigma_1 \circ_{lin} \sigma_2)(f), (\sigma_1 \circ_{lin} \sigma_2)^{\frown}[x_{\pi(1)}], \ldots, (\sigma_1 \circ_{lin} \sigma_2)^{\frown}[x_{\pi(n)}]) \\
&= S^n_{lin}((\widehat{\sigma}_1 \circ \sigma_2)(f), (\widehat{\sigma}_1 \circ \widehat{\sigma}_2)[x_{\pi(1)}], \ldots, (\widehat{\sigma}_1 \circ \widehat{\sigma}_2)[x_{\pi(n)}]) \\
&= S^n_{lin}(\widehat{\sigma}_1[\sigma_2(f)], \widehat{\sigma}_1[\widehat{\sigma}_2[x_{\pi(1)}]], \ldots, \widehat{\sigma}_1[\widehat{\sigma}_2[x_{\pi(n)}]]) \\
&= \widehat{\sigma}_1[S^n_{lin}(\sigma_2(f), \widehat{\sigma}_2[x_{\pi(1)}], \ldots, \widehat{\sigma}_2[x_{\pi(n)}])] \\
&= \widehat{\sigma}_1[\widehat{\sigma}_2[f(x_{\pi(1)}, \ldots, x_{\pi(n)})]] \\
&= (\widehat{\sigma}_1 \circ \widehat{\sigma}_2)[f(x_{\pi(1)}, \ldots, x_{\pi(n)})].
\end{aligned}$$

For $F \in \mathcal{F}^{lin}_{((n);(n))}(X_n)$, we will give a proof by induction on the complexity of the definition of a linear-formula F.

(i) If F has the form $x_i \approx x_j$ for $i \ne j \in \{1, \ldots, n\}$,
then $(\sigma_1 \circ_{lin} \sigma_2)^{\frown}[x_i \approx x_j]$

$$\begin{aligned}
&= (\sigma_1 \circ_{lin} \sigma_2)^{\frown}[x_i] \approx (\sigma_1 \circ_{lin} \sigma_2)^{\frown}[x_j] \\
&= x_i \approx x_j. \\
&= \widehat{\sigma}_1[\widehat{\sigma}_2[x_i]] \approx \widehat{\sigma}_1[\widehat{\sigma}_2[x_j]] \\
&= (\widehat{\sigma}_1 \circ \widehat{\sigma}_2)[x_i] \approx (\widehat{\sigma}_1 \circ \widehat{\sigma}_2)[x_j] \\
&= (\widehat{\sigma}_1 \circ \widehat{\sigma}_2)[x_i \approx x_j].
\end{aligned}$$

(ii) If F has the form $\gamma(x_{\pi(1)}, \ldots, x_{\pi(n)})$, then
$(\sigma_1 \circ_{lin} \sigma_2)^{\frown}[\gamma(x_{\pi(1)}, \ldots, x_{\pi(n)})]$

$$\begin{aligned}
&= R^n_{lin}((\sigma_1 \circ_{lin} \sigma_2)(\gamma), (\sigma_1 \circ_{lin} \sigma_2)^{\frown}[x_{\pi(1)}], \ldots, (\sigma_1 \circ_{lin} \sigma_2)^{\frown}[x_{\pi(n)}]) \\
&= R^n_{lin}((\widehat{\sigma}_1 \circ \sigma_2)(\gamma), x_{\pi(1)}, \ldots, x_{\pi(n)}) \\
&= R^n_{lin}(\widehat{\sigma}_1[\sigma_2(\gamma)], x_{\pi(1)}, \ldots, x_{\pi(n)}) \\
&= \widehat{\sigma}_1[\widehat{\sigma}_2[\gamma(x_{\pi(1)}, \ldots, x_{\pi(n)})]] \\
&= (\widehat{\sigma}_1 \circ \widehat{\sigma}_2)[\gamma(x_{\pi(1)}, \ldots, x_{\pi(n)})].
\end{aligned}$$

(iii) If F has the form $\neg F$ and assume that $(\sigma_1 \circ_{lin} \sigma_2)^{\frown}[F] = (\widehat{\sigma}_1 \circ \widehat{\sigma}_2)[F]$, then
$(\sigma_1 \circ_{lin} \sigma_2)^{\frown}[\neg F] = \neg((\sigma_1 \circ_{lin} \sigma_2)^{\frown}[F]) = \neg((\widehat{\sigma}_1 \circ \widehat{\sigma}_2)[F]) = \widehat{\sigma}_1[\neg(\widehat{\sigma}_2[F])] = \widehat{\sigma}_1[\widehat{\sigma}_2[\neg(F)]] = (\widehat{\sigma}_1 \circ \widehat{\sigma}_2)[\neg(F)]$. $\square$

Let σ_{id} be the linear-hypersubstitution for algebraic systems of type $((n);(n))$ which maps the operation symbols f to the linear-term $f(x_1, \ldots, x_n)$, and the relational symbols γ to the linear-formula $\gamma(x_1, \ldots, x_n)$.

Lemma 4.4. *Let $n \in \mathbb{N}$. For any $t \in W^{lin}_n(X_n)$ and any $F \in \mathcal{F}^{lin}_{((n);(n))}(X_n)$. We have*

$$\widehat{\sigma}_{id}[t] = t \text{ and } \widehat{\sigma}_{id}[F] = F.$$

Proof. Let $t \in W^{lin}_n(X_n)$, we will give a proof by induction on the definition of a linear-term t.

(i) If $t = x_i$; $i \in 1 \le i \le n$, then $\widehat{\sigma}_{id}[x_i] = x_i$.

(ii) If $t = f(x_{\pi(1)}, \ldots, x_{\pi(n)}), \pi \in P_n$, then $\widehat{\sigma}_{id}[f(x_{\pi(1)}, \ldots, x_{\pi(n)})]$

$$\begin{aligned}
&= S^n_{lin}(\sigma_{id}(f), \widehat{\sigma}_{id}[x_{\pi(1)}], \ldots, \widehat{\sigma}_{id}[x_{\pi(n)}]) \\
&= S^n_{lin}(f(x_1, \ldots, x_n), x_{\pi(1)}, \ldots, x_{\pi(n)})\ ) \\
&= f(S^n_{lin}(x_1, x_{\pi(1)}, \ldots, x_{\pi(n)}), \ldots, S^n_{lin}(x_n, x_{\pi(1)}, \ldots, x_{\pi(n)})) \\
&= f(x_{\pi(1)}, \ldots, x_{\pi(n)}).
\end{aligned}$$

For $F \in \mathcal{F}^{lin}_{((n);(n))}(X_n)$, we will give a proof by induction on the definition of a linear-formula F.
(i) If F has the form $x_i \approx x_j$ for $i \neq j \in \{1, \ldots, n\}$, then $\widehat{\sigma}_{id}[x_i \approx x_j] = \widehat{\sigma}_{id}[x_i] \approx \widehat{\sigma}_{id}[x_j] = x_i \approx x_j$.
(ii) If F has the form $\gamma(x_{\pi(1)}, \ldots, x_{\pi(n)})$, then

$$\begin{aligned}
&\widehat{\sigma}_{id}[\gamma(x_{\pi(1)}, \ldots, x_{\pi(n)})]\\
&= R^n_{lin}(\sigma_{id}(\gamma), \widehat{\sigma}_{id}[x_{\pi(1)}], \ldots, \widehat{\sigma}_{id}[x_{\pi(n)}]\\
&= R^n_{lin}(\gamma(x_1, \ldots, x_n), x_{\pi(1)}, \ldots, x_{\pi(n)})\\
&= \gamma(S^n_{lin}(x_1, x_{\pi(1)}, \ldots, x_{\pi(n)}), \ldots, S^n_{lin}(x_n, x_{\pi(1)}, \ldots, x_{\pi(n)}))\\
&= \gamma(x_{\pi(1)}, \ldots, x_{\pi(n)}).
\end{aligned}$$

(iii) If F has the form $\neg F$ and assume that $\widehat{\sigma}_{id}[F] = F$, then $\widehat{\sigma}_{id}[\neg F] = \neg\widehat{\sigma}_{id}[F] = \neg F$. □

Theorem 4.5. *$\mathcal{H}yp^{lin}((n);(n)) := (Hyp^{lin}((n);(n)); \circ_{lin}, \sigma_{id})$ is a monoid.*

Proof. Using Lemma 4.3 and using the fact that $\circ$ is associative, it can be shown that $\circ_{lin}$ is associative. In fact, for every $\sigma_1, \sigma_2, \sigma_3 \in Hyp^{lin}((n);(n))$ we have

$$\begin{aligned}
\sigma_1 \circ_{lin} (\sigma_2 \circ_{lin} \sigma_3) &= \widehat{\sigma}_1 \circ (\sigma_2 \circ_{lin} \sigma_3) = \widehat{\sigma}_1 \circ (\widehat{\sigma}_2 \circ \sigma_3) = (\widehat{\sigma}_1 \circ \widehat{\sigma}_2) \circ \sigma_3\\
&= (\sigma_1 \circ_{lin} \sigma_2)\widehat{} \circ \sigma_3 = (\sigma_1 \circ_{lin} \sigma_2) \circ_{lin} \sigma_3.
\end{aligned}$$

Using Lemma 4.4 shows that σ_{id} is an identity element with respect to $\circ_{lin}$. First, we will show that σ_{id} is left identity element. Let $\beta \in \{f\} \cup \{\gamma\}$, then $(\sigma_{id} \circ_{lin} \sigma)(\beta) = (\widehat{\sigma}_{id} \circ \sigma)(\beta) = \widehat{\sigma}_{id}[\sigma(\beta)] = \sigma(\beta)$.
Now, we will show that σ_{id} is a right identity element as follows:
If $\beta = f$, then

$$\begin{aligned}
(\sigma \circ_{lin} \sigma_{id})(f) &= (\widehat{\sigma} \circ \sigma_{id})(f) = \widehat{\sigma}[\sigma_{id}(f)] = \widehat{\sigma}[f(x_1, \ldots, x_n)]\\
&= S^n_{lin}(\sigma(f), \widehat{\sigma}[x_1], \ldots, \widehat{\sigma}[x_n]) = S^n_{lin}(\sigma(f), x_1, \ldots, x_n) = \sigma(f).
\end{aligned}$$

If $\beta = \gamma$, then

$$\begin{aligned}
(\sigma \circ_{lin} \sigma_{id})(\gamma) &= (\widehat{\sigma} \circ \sigma_{id})(\gamma) = \widehat{\sigma}[\sigma_{id}(\gamma)] = \widehat{\sigma}[\gamma(x_1, \ldots, x_n)]\\
&= R^n_{lin}(\sigma(\gamma), \widehat{\sigma}[x_1], \ldots, \widehat{\sigma}[x_n]) = \sigma(\gamma).
\end{aligned}$$

Therefore $\sigma_{id} \circ_{lin} \sigma = \sigma = \sigma \circ_{lin} \sigma_{id}$. □

5 All Idempotent Elements of Linear-Hypersubstitutions for Algebraic Systems of type ((n);(n))

In this section, we will characterize all idempotent elements of linear-hypersubstitutions for algebraic systems of type $((n);(n))$. A linear-hypersubstitutions σ for algebraic systems which map f to a linear-term t and γ to a linear-formula F preserves arities is denoted by $\sigma := \sigma_{t,F}$ that means $\sigma_{t,F}(f) = t$ and $\sigma_{t,F}(\gamma) = F$. First, we will recall the definition of an idempotent element.

Definition 5.1. *[3] Let $(S;\cdot)$ be a semigroup and $a \in S$ is called idempotent element if $a \cdot a = a$. In general, we denote the set of all idempotent elements of S by $E(S)$.*

Proposition 5.2. *For any $t \in W^{lin}_n(X_n)$ and $F \in \mathcal{F}^{lin}_{((n);(n))}(X_n)$. The element $\sigma_{t,F} \in Hyp^{lin}((n);(n))$ is an idempotent if and only if $\widehat{\sigma}_{t,F}[t] = t$ and $\widehat{\sigma}_{t,F}[F] = F$.*

Proof. Assume that $\sigma_{t,F}$ is an idempotent, i.e. $(\sigma_{t,F} \circ_{lin} \sigma_{t,F})(f) = \sigma_{t,F}(f)$ and $(\sigma_{t,F} \circ_{lin} \sigma_{t,F})(\gamma) = \sigma_{t,F}(\gamma)$. Then $\widehat{\sigma}_{t,F}[t] = \widehat{\sigma}_{t,F}[\sigma_{t,F}(f)] = (\sigma_{t,F} \circ_{lin} \sigma_{t,F})(f) = \sigma_{t,F}(f) = t$ and $\widehat{\sigma}_{t,F}[F] = \widehat{\sigma}_{t,F}[\sigma_{t,F}(\gamma)] = (\sigma_{t,F} \circ_{lin} \sigma_{t,F})(\gamma) = \sigma_{t,F}(\gamma) = F$. Conversely, let $\widehat{\sigma}_{t,F}[t] = t$ and $\widehat{\sigma}_{t,F}[F] = F$, we have $(\sigma_{t,F} \circ_{lin} \sigma_{t,F})(f) = \widehat{\sigma}_{t,F}[\sigma_{t,F}(f)] = \widehat{\sigma}_{t,F}[t] = t = \sigma_{t,F}(f)$ and $(\sigma_{t,F} \circ_{lin} \sigma_{t,F})(\gamma) = \widehat{\sigma}_{t,F}[\sigma_{t,F}(\gamma)] = \widehat{\sigma}_{t,F}[F] = F = \sigma_{t,F}(\gamma)$. This shows that $\sigma_{t,F}$ is an idempotent element. □

Proposition 5.3. *If $t = x \in X_n$ and $F = x_l \approx x_k$ for $l \neq k \in \{1, \ldots, n\}$, then $\sigma_{t,F} \in Hyp^{lin}((n);(n))$ is an idempotent element.*

Proof. Let $\sigma_{t,F} \in Hyp^{lin}((n);(n))$ for every $n \in \mathbb{N}$, let $t = x \in X_n$ and $F = x_l \approx x_k$ for $l \neq k \in \{1,\ldots,n\}$. We have $\widehat{\sigma}_{t,F}[x] = x = t$ and $\widehat{\sigma}_{t,F}[x_l \approx x_k] = \widehat{\sigma}_{t,F}[x_l] \approx \widehat{\sigma}_{t,F}[x_k] = x_l \approx x_k = F$.
By Proposition 5.2, $\sigma_{t,F}$ is an idempotent element. □

Proposition 5.4. *If $t = x \in X_n$ and $F = \gamma(x_1,\ldots,x_n)$, then $\sigma_{t,F} \in Hyp^{lin}((n);(n))$ is an idempotent element.*

Proof. Let $\sigma_{t,F} \in Hyp^{lin}((n);(n))$ for every $n \in \mathbb{N}$, let $t = x \in X_n$ and $F = \gamma(x_1,\ldots,x_n)$. We have $\widehat{\sigma}_{t,F}[x] = x$ and

$$\begin{aligned}
\widehat{\sigma}_{t,F}[\gamma(x_1,\ldots,x_n)] &= R^n_{lin}(\sigma_{t,F}(\gamma),\widehat{\sigma}_{t,F}[x_1],\ldots,\widehat{\sigma}_{t,F}[x_n]) \\
&= R^n_{lin}(\gamma(x_1,\ldots,x_n),x_1,\ldots,x_n) \\
&= \gamma(S^n_{lin}(x_1,x_1,\ldots,x_n),\ldots,S^n_{lin}(x_n,x_1,\ldots,x_n)) \\
&= \gamma(x_1,\ldots,x_n).
\end{aligned}$$

By Proposition 5.2, $\sigma_{t,F}$ is an idempotent element. □

Proposition 5.5. *If $t = f(x_1,\ldots,x_n)$ and $F = x_l \approx x_k$, for $l \neq k \in \{1,\ldots,n\}$, then $\sigma_{t,F} \in Hyp^{lin}((n);(n))$ is an idempotent element.*

Proof. Let $\sigma_{t,F} \in Hyp^{lin}((n);(n))$. For every $n \in \mathbb{N}$, let $t = f(x_1,\ldots,x_n)$ and $F = x_l \approx x_k$, for $l \neq k \in \{1,\ldots,n\}$. We have

$$\begin{aligned}
\widehat{\sigma}_{t,F}[f(x_1,\ldots,x_n)] &= S^n_{lin}(\sigma_{t,F}(f),\widehat{\sigma}_{t,F}[x_1],\ldots,\widehat{\sigma}_{t,F}[x_n]) \\
&= S^n_{lin}(f(x_1,\ldots,x_n),x_1,\ldots,x_n) \\
&= f(S^n_{lin}(x_1,x_1,\ldots,x_n),\ldots,S^n_{lin}(x_n,x_1,\ldots,x_n) \\
&= f(x_1,\ldots,x_n).
\end{aligned}$$

By Proposition 5.3, we get $\widehat{\sigma}_{t,F}[x_l \approx x_k] = x_l \approx x_k$. Therefore $\sigma_{t,F}$ is an idempotent element. □

Proposition 5.6. *If $t = f(x_1,\ldots,x_n)$ and $F = \gamma(x_1,\ldots,x_n)$, then $\sigma_{t,F} \in Hyp^{lin}((n);(n))$ is an idempotent element.*

Proof. In a similar way to the proof of Proposition 5.3 and Proposition 5.4, we proceed for $\widehat{\sigma}_{t,F}[f(x_1,\ldots,x_n)]$ $= f(x_1,\ldots,x_n)$ and $\widehat{\sigma}_{t,F}[\gamma(x_1,\ldots,x_n)] = \gamma(x_1,\ldots,x_n)$, respectively □

Proposition 5.7. *Let $\sigma_{t,F} \in Hyp^{lin}(((n);(n))$. If $\widehat{\sigma}_{t,F}[t] = t$ and $F = x_l \approx x_k$ for $l \neq k \in \{1,\ldots,n\}$, then $\sigma_{t,\neg F}$ is an idempotent element.*

Proof. Let $\sigma_{t,F} \in Hyp^{lin}(((n);(n))$. For $\widehat{\sigma}_{t,F}[t] = t$ and $F = x_l \approx x_k$ for $l \neq k \in \{1,\ldots,n\}$, we get $(\sigma_{t,\neg F} \circ_{lin} \sigma_{t,\neg F})(f) = \widehat{\sigma}_{t,\neg F}[\sigma_{t,\neg F}(f)] = \widehat{\sigma}_{t,\neg F}[t] = t = \sigma_{t,\neg F}(f)$ and $(\sigma_{t,\neg F} \circ_{lin} \sigma_{t,\neg F})(\gamma)$ $= \widehat{\sigma}_{t,\neg F}[\sigma_{t,\neg F}(\gamma)] = \widehat{\sigma}_{t,\neg F}[\neg F] = \neg(\widehat{\sigma}_{t,\neg F}[F]) = \neg F = \sigma_{t,\neg F}(\gamma)$. □

If ρ is a permutation on set $\{1,2,\ldots,n\}$ such that ρ replaces each element by the element itself, ρ is called the identity permutation on set $\{1,2,\ldots,n\}$. Thus

$$\rho = \begin{pmatrix} 1 & 2 & 3 & \ldots & n \\ 1 & 2 & 3 & \ldots & n \end{pmatrix}.$$

Proposition 5.8. *Let $n \in \mathbb{N}$ and ρ be an identity permutation on the set $\{1,2,\ldots,n\}$. If $t = f(x_{\pi(1)},\ldots,x_{\pi(n)})$ or $F = \gamma(x_{\pi(1)},\ldots,x_{\pi(n)})$ where π is a permutation such that $\pi \neq \rho$, then $\sigma_{t,F} \in Hyp^{lin}((n);(n))$ is not an idempotent element.*

Proof. If $t = f(x_{\pi(1)},\ldots,x_{\pi(n)})$, then

$$\begin{aligned}
(\sigma_{t,F} \circ_{lin} \sigma_{t,F})(f) &= \widehat{\sigma}_{t,F}[\sigma_{t,F}(f)] = \widehat{\sigma}_{t,F}[f(x_{\pi(1)},\ldots,x_{\pi(n)})] \\
&= S^n_{lin}(\sigma_{t,F}(f),\widehat{\sigma}_{t,F}[x_{\pi(1)}],\ldots,\widehat{\sigma}_{t,F}[x_{\pi(n)}] \\
&= S^n_{lin}(f(x_{\pi(1)},\ldots,x_{\pi(n)}),x_{\pi(1)},\ldots,x_{\pi(n)}) \\
&= f(S^n_{lin}(x_{\pi(1)},x_{\pi(1)},\ldots,x_{\pi(n)}),\ldots,S^n_{lin}(x_{\pi(n)},x_{\pi(1)},\ldots,x_{\pi(n)}). \\
&\neq f(x_{\pi(1)},\ldots,x_{\pi(n)}) \qquad (\because \pi \neq \rho) \\
&\neq \sigma_{t,F})(f).
\end{aligned}$$

If $F = \gamma(x_{\pi(1)}, \ldots, x_{\pi(n)})$, then

$$\begin{aligned}(\sigma_{\iota,F} \circ_{lin} \sigma_{\iota,F})(\gamma) &= \widehat{\sigma}_{\iota,F}[\sigma_{\iota,F}(\gamma)] = \widehat{\sigma}_{\iota,F}[\gamma(x_{\pi(1)}, \ldots, x_{\pi(n)})] \\ &= R^n_{lin}(\sigma_{\iota,F}(\gamma), \widehat{\sigma}_{\iota,F}[x_{\pi(1)}], \ldots, \widehat{\sigma}_{\iota,F}[x_{\pi(n)}] \\ &= R^n_{lin}(\gamma(x_{\pi(1)}, \ldots, x_{\pi(n)}), x_{\pi(1)}, \ldots, x_{\pi(n)}) \\ &= \gamma(S^n_{lin}(x_{\pi(1)}, x_{\pi(1)}, \ldots, x_{\pi(n)}), \ldots, S^n_{lin}(x_{\pi(n)}, x_{\pi(1)}, \ldots, x_{\pi(n)}). \\ &\neq \gamma(x_{\pi(1)}, \ldots, x_{\pi(n)}) \qquad (\because \pi \neq \rho) \\ &\neq \sigma_{\iota,F})(\gamma).\end{aligned}$$

Therefore $\sigma_{\iota,F}$ is not an idempotent element. □

Proposition 5.9. *Let $t = x \in X_n$ and $F = \gamma(x_{\pi(1)}, \ldots, x_{\pi(n)})$ whenever $\pi \neq \rho$, then $\sigma_{\iota,F} \in Hyp^{lin}((n);(n))$ is not an idempotent element.*

Proof. It is an immediate consequence of Proposition 5.8. □

6 Conclusion

The main result of the paper is the characterization of idempotent elements of linear-hypersubstitutions for algebraic systems of type $((n);(n))$. We can check that all these linear-hypersubstitutions for algebraic systems of type $((n);(n))$ which satisfy the conditions are idempotent by using Proposition 5.3-5.7.

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